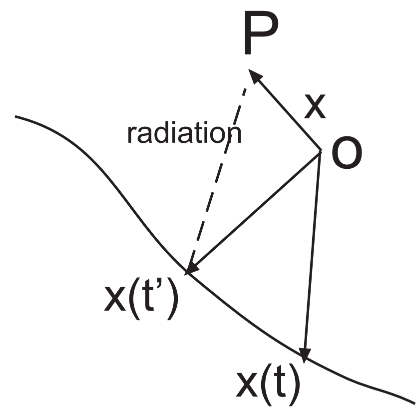
Northern Illinois University, PHY 571, Fall 2006

Part IV: Radiation from accelerating charges

Last updated on November 14, 2006 (report errors to piot@fnal.gov)

Radiation from accelerating charges



Radiation emitted at time t' reaches the observer (P) at time $t > t'$. It is retarded due to the finite speed of light. Let's first derive the 4-potential due to the moving charge.

Four-potential produced by a moving charge

Start with the inhomogenous Maxwell's equation:

$$\partial_\alpha = F^{\alpha\beta} = \frac{4\pi}{c} j^\beta \tag{1}$$

j^β is the 4-current $j^\beta \equiv (c\rho, \overrightarrow{J})$. Use the definition of $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ and impose the Lorenz Gauge condition $\partial_\alpha A^\alpha = 0$ we have:

$$\partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi j^\beta}{c} \tag{2}$$

which can be re-written:

$$\Box A^\beta = \frac{4\pi}{c} j^\beta(x) \tag{3}$$

Solution of the latter equation $\rightarrow$ find Green's function $D(x, x')$ for the equation

$$\Box_x D(x, x') = \delta^{(4)}(x - x') \tag{4}$$

where $\delta^{(4)}(x-x') \equiv \delta(x_0-x'_0)\delta^{(3)}(\overrightarrow{x}-\overrightarrow{x'})$. If free-space (no boundary condition) then $D(x,x') = D(x-x')$. Let $z^\alpha = x^\alpha - x'^\alpha$, $D(x-x') \rightarrow D(z)$ and the d'Alembert equation rewrites:

$$\Box_z D(z) = \delta^{(4)}(z).$$

Which can be solved using the Fourier transform method: write

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^4}\int d^4k\tilde{D}(k)e^{-ikz}, \text{ and,} \\ \delta^4(z) &= \frac{1}{(2\pi)^4}\int d^4k e^{-ikz} \end{aligned} \tag{5}$$

Expliciting in the wave equation on finds:

$$\tilde{D}(k) = -\frac{1}{k_\beta k^\beta}$$

where $k^\beta \equiv (k_0, \overrightarrow{\kappa})$ is the four-wavevector, and let $z = (z_0, \overrightarrow{R})$. $k_\beta k^\beta = k_0^2 - \kappa^2$.

so the Green function is given by:

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^4} \int d^4k(-) \frac{e^{-ikz}}{k_0^2 - \kappa^2} \\ &= -\frac{1}{(2\pi)^4} \int d^3\kappa e^{i\vec{\kappa}\vec{R}} \int dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} \end{aligned} \tag{6}$$

Consider the integral over k_0. It can be replaced by an integral over a closed contour in the complex space associated to k_0. The integrand has two poles at $k_0 \pm \kappa$ on the real axis. If we consider $z_0 > 0$ the contour need to be closed toward $\mathcal{I}m(k_0) = -\infty$ and the integral is:

$$\begin{aligned} \int_{\infty}^{+\infty} dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} &= \oint_C dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} = -2i\pi \sum \text{Res}\left(\frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}\right) \\ &= -\frac{2\pi}{\kappa} \sin(kz_0) \end{aligned} \tag{7}$$

So D, the retarded Green function, becomes:

$$D(z) = \frac{1}{(2\pi)^3}\int d^3k \frac{\sin(\kappa z_0)}{\kappa} e^{i\overrightarrow{\kappa}.\overrightarrow{R}} \ (z_0 > 0) \tag{8}$$
$$= \frac{\Theta(z_0)}{(2\pi)^3}\int d^3k \frac{\sin(\kappa z_0)}{\kappa} e^{i\overrightarrow{\kappa}.\overrightarrow{R}}$$

where $\Theta(x)$ is the Heaviside function. Introducing $d^3\kappa = k^2 d\kappa \sin(\theta) d\theta d\phi$ then we can work out the integral over angle:

$$\int \sin\theta d\theta d\phi e^{i\overrightarrow{\kappa}\overrightarrow{R}} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta e^{i\kappa z\cos\theta} = 2\pi \left[\frac{e^{i\kappa z\cos\theta}}{-i\kappa z}\right]_0^{\pi}$$
$$= 4\pi \frac{\sin(\kappa R)}{\kappa R}. \tag{9}$$

So,

$$
\begin{aligned}
D(z) &= \frac{\Theta(z_0)}{(2\pi)^3} \int d\kappa \frac{4\pi}{R} \sin(\kappa R) \sin(\kappa z_0) && (10) \\
&= \frac{\Theta(z_0)}{2\pi^2 R} \int_0^{\infty} d\kappa \sin(\kappa R) \sin(\kappa z_0) \\
&= -\frac{1}{4\pi R} \frac{1}{2\pi} \int_0^{+\infty} \left[e^{ik(R+z_0)} - e^{ik(R-z_0)} - e^{-ik(R-z_0)} + e^{ik(R+z_0)} \right] \\
&= \frac{\Theta(z_0)}{4\pi R} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[-e^{ik(R+z_0)} - e^{ik(R-z_0)} \right] && (11) \\
&= \frac{\Theta(z_0)}{4\pi R} \left[\delta(z_0 - R) + \delta(z_0 + R) \right] = \frac{\Theta(z_0)}{4\pi R} \delta(z_0 - R) && (12)
\end{aligned}
$$

since the condition $z_0 > 0$ implies $\delta(z_0 + R) = 0$.

$$D(x - x') = \frac{\Theta(z_0)}{4\pi R}\delta(x - x' - R) \tag{13}$$

Now use the identity

$$\begin{aligned}
\delta[(x - x')^2] &= \delta[(x - x_0)^2 - |x - x'|^2] \\
&= \delta[(x_0 - x'_0 - R)(x_0 - x'_0 + R)] \\
&= \frac{1}{2R}\left[\delta(x_0 - x'_0 - R) + \delta(x_0 - x'_0 + R)\right]
\end{aligned} \tag{14}$$

where we make the use of $\delta[(x-x_1)(x-x_2)] = \frac{\delta(x-x_1)+\delta(x-x_2)}{|x_1-x_2|}$. The function D becomes:

$$D(x - x') = \frac{1}{2\pi}\Theta(x_0 - x'_0)\delta[(x - x')^2]. \tag{15}$$

Then the retarded 4-potential is given by the convolution integral:

$$A^\alpha(x) = \text{const.} + \frac{4\pi}{c}\int d^4x' D(x - x')J^\alpha(x') \tag{16}$$

Liénard-Wiechert Potentials

The 4-potential caused by a charge in motion is:

$$A^{\alpha}(x) = \frac{4\pi}{c}\int d^4x' D(x-x')j^{\alpha}(x'), \tag{17}$$

The 4-current is (see Part **II**)

$$j^{\alpha}(x') = ec\int d\tau v^{\alpha}(\tau)\delta^{(4)}[x'-r(\tau)] \tag{18}$$

τ is the charge's proper time. So expliciting D and j^{α} the 4-potential takes the form

$$\begin{aligned} A^{\alpha}(x) &= 2e\int d\tau d^4x' \Theta(x_0 - x'_0)\delta[(x-x')^2]v^{\alpha}(\tau)\delta^{(4)}[x-r(\tau)] \\ &= 2e\int d\tau \Theta(x_0 - x'_0)v^{\alpha}(\tau)\delta[(x-r(\tau))^2] \end{aligned} \tag{19}$$

$$A^\alpha(x) \;=\; 2e\int d\tau\delta(\tau-\tau_0)\Theta(x_0-x'_0)v^\alpha(\tau)\left|\frac{-1}{2v^\beta(\tau)[x-r(\tau)]_\beta}\right|$$

using the relation

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{\left|\frac{\partial f}{\partial x}\right|_{x=x_i}}.$$

The four-vector potential finally writes:

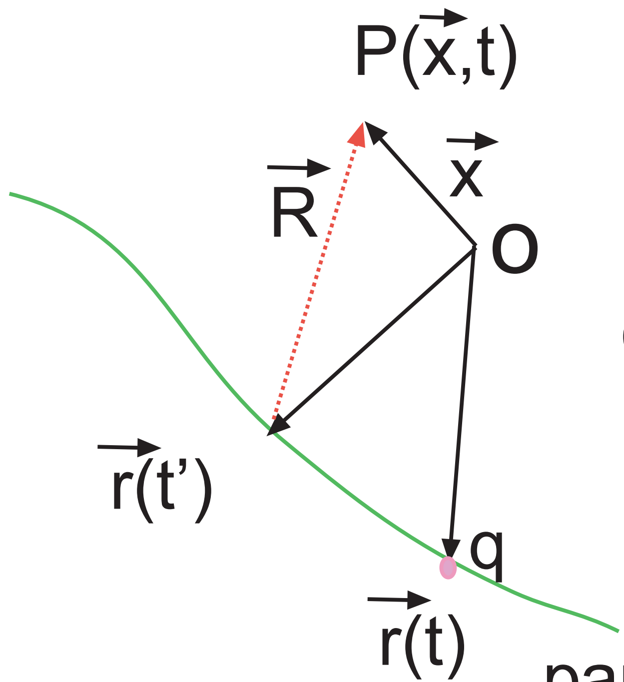
$$A^\alpha(x) = \left.\frac{ev^\alpha(\tau)}{v^\beta[x-r(\tau)]_\beta}\right|_{\tau=\tau_0}$$

which can be written in the more familiar form:

$$\Phi(\overrightarrow{x},t) = \left[\frac{e}{(1-\overrightarrow{\beta}.\widehat{n})R}\right]_{ret}, \text{ and, } \overrightarrow{A}(\overrightarrow{x},t) = \left[\frac{e\overrightarrow{\beta}}{(1-\overrightarrow{\beta}.\widehat{n})R}\right]_{ret} \quad (20)$$

where ret means the quantity in bracket have to be evaluated at the retarded time t' that satisfies the causality condition.

Field associated to a moving charge



Causality imposes:

$c(t-t')=|\vec{R}|$

Field associated to a moving charge

Consider a charge q in motion the Lienard-Witchert potential are given by:

$$\begin{pmatrix} \Phi(\overrightarrow{x},t) \\ \overrightarrow{A}(\overrightarrow{x},t) \end{pmatrix} = \left[\frac{q}{(1-\overrightarrow{\beta}.\widehat{n})R}\begin{pmatrix} 1 \\ \overrightarrow{\beta} \end{pmatrix}\right]_{ret}. \tag{21}$$

The fields are given by $\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t}$, but, we need to evaluate the quantities at the retarded time t'. **First** let's express the $\overrightarrow{\nabla}$ and $\partial/\partial t$ operators in term of retarded quantities.

$$R(t') = c(t-t') \Rightarrow \frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right). \tag{22}$$

On another hand:

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t}. \tag{23}$$

$$\frac{1}{2}\frac{\partial R^2}{\partial t'} = R\frac{\partial R}{\partial t'} = \overrightarrow{R}.\frac{\partial \overrightarrow{R}}{\partial t'} \ , \ \ \text{so,} \ R\frac{\partial R}{\partial t'} = -\overrightarrow{v}.\overrightarrow{R}. \tag{24}$$

Thus

$$\frac{\partial R}{\partial t} = -c\overrightarrow{\beta}.\hat{n}\frac{\partial t'}{\partial t} \tag{25}$$

From equation 22 and 25 one gets:

$$\frac{\partial t'}{\partial t} = \frac{1}{1-\overrightarrow{\beta}.\hat{n}} \equiv \frac{1}{\kappa} \quad \Rightarrow \frac{\partial}{\partial t} = \frac{1}{\kappa}\frac{\partial}{\partial t'} \tag{26}$$

For the operator $\overrightarrow{\nabla}$ take:

$$\overrightarrow{\nabla}R = \overrightarrow{\nabla}[c(t-t')] = -c\overrightarrow{\nabla}t' \tag{27}$$

if $\overrightarrow{\nabla}_{t'}$ is the gradient operator evaluated at constant t' then

$$\begin{aligned} \overrightarrow{\nabla}R &= \overrightarrow{\nabla}_{t'}R + \frac{\partial R}{\partial t'}\overrightarrow{\nabla}t' \\ &= \hat{n} - c\overrightarrow{\beta}.\hat{n}\overrightarrow{\nabla}t' \end{aligned} \tag{28}$$

From equation 27 and 28 one gets:

$$\overrightarrow{\nabla} t' = \frac{-\hat{n}}{c(1 - \overrightarrow{\beta}.\hat{n})} \tag{29}$$

So we finally get:

$$\overrightarrow{\nabla} = \overrightarrow{\nabla}_{t'} - \frac{\hat{n}}{c\kappa}\frac{\partial}{\partial t'} \tag{30}$$

So the electric field is :

$$\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t} = -\overrightarrow{\nabla}_{t'}\Phi + \frac{\hat{n}}{c\kappa}\frac{\partial \Phi}{\partial t'} - \frac{1}{\kappa c}\frac{\partial \overrightarrow{A}}{\partial t'}, \tag{31}$$

with $\Phi = \frac{e}{\kappa R}$.

$$\overrightarrow{\nabla}_{t'}\Phi = \frac{-e}{(\kappa R)^2}[R\overrightarrow{\nabla}_{t'}\kappa + \kappa\overrightarrow{\nabla}_{t'}R] \tag{32}$$

$\overrightarrow{\nabla}_{t'}R = \hat{n}$, and

$$\overrightarrow{\nabla}_{t'}\kappa = \overrightarrow{\nabla}_{t'}(1 - \overrightarrow{\beta}.\hat{n}) = -\overrightarrow{\nabla}_{t'}(\overrightarrow{\beta}.\hat{n}) = -(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'})\hat{n}. \tag{33}$$

$$
\begin{aligned}
\Rightarrow \overrightarrow{\nabla}_{t'}\kappa &= -(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'})\frac{\overrightarrow{R}}{R} \\
&= -\frac{R(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'})\overrightarrow{R} - \overrightarrow{R}(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'}}{R^2} \\
&= -\frac{\overrightarrow{\beta} - \hat{n}(\overrightarrow{\beta}.\hat{n})}{R}.
\end{aligned} \tag{34}
$$

So we finally get:

$$
\begin{aligned}
\overrightarrow{\nabla}_{t'}\Phi &= \frac{-e}{(\kappa R)^2}[-\overrightarrow{\beta} + \hat{n}(\overrightarrow{\beta}.\hat{n}) + (1 - \overrightarrow{\beta}.\hat{n})\hat{n}] \\
&= -\frac{e}{(\kappa R)^2}[\hat{n} - \overrightarrow{\beta}]
\end{aligned} \tag{35}
$$

where we have used $\kappa = 1 - \overrightarrow{\beta}.\hat{n}$. Now let's calculate the quantity $\frac{\hat{n}}{c\kappa}\frac{\partial}{\partial t'}\Phi$:

$$
\frac{\partial \Phi}{\partial t'} = -e\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right) = \frac{-e}{(\kappa R)^2}[\kappa \dot{R} + R\dot{\kappa}] \tag{36}
$$

$\dot{R} = -c\overrightarrow{\beta}.\widehat{n}$, and $\dot{\kappa} = -\overrightarrow{\dot{\beta}}.\widehat{n} - \dot{\widehat{n}}.\overrightarrow{\beta}$.

$$\begin{aligned} \dot{\widehat{n}} &= \frac{\partial}{\partial t'}\frac{\overrightarrow{R}}{R} = \frac{R\overrightarrow{\dot{R}} - \dot{R}\overrightarrow{R}}{R^2} \\ &= \frac{-\overrightarrow{v} + (\overrightarrow{v}.\widehat{n})\widehat{n}}{R} = -c\frac{\overrightarrow{\beta} - (\overrightarrow{\beta}.\widehat{n})\widehat{n}}{R}. \end{aligned} \tag{37}$$

Then

$$\begin{aligned} \kappa\dot{R} + \dot{\kappa}R &= -c\overrightarrow{\beta}.\widehat{n}\kappa + R\{-\overrightarrow{\dot{\beta}}.\widehat{n} + c\overrightarrow{\beta}[\frac{\overrightarrow{\beta} - (\overrightarrow{\beta}.\widehat{n})\widehat{n}}{R}]\} \\ &= -\overrightarrow{\dot{\beta}}.\overrightarrow{R} + c\beta^2 - c\overrightarrow{\beta}.\widehat{n} \end{aligned} \tag{38}$$

So

$$\dot{\Phi} = -\frac{e}{(\kappa R)^2}[-\overrightarrow{\dot{\beta}}.\overrightarrow{R} + c\beta^2 - c\overrightarrow{\beta}.\widehat{n}]. \tag{39}$$

So from Equations 35 and 39 we have

$$\overrightarrow{\nabla}_{t'}\Phi = -\frac{e}{(\kappa R)^2}\{\widehat{n} - \overrightarrow{\beta} + \frac{\widehat{n}}{c\kappa}[+\overrightarrow{\dot{\beta}}.\overrightarrow{R} - c\beta^2 + c\overrightarrow{\beta}.\widehat{n}]\} \tag{40}$$

Now we need to compute $\frac{\partial \vec{A}}{\partial t}$:

$$\frac{\partial \vec{A}}{\partial t} = \vec{\dot{A}} \frac{\partial t'}{\partial t} = \frac{1}{\kappa} \vec{\dot{A}} \tag{41}$$

So

$$\vec{\dot{A}} = \vec{\dot{\beta}} \Phi + \vec{\beta} \dot{\Phi} = + \frac{e}{R\kappa} \vec{\dot{\beta}} \frac{e\vec{\beta}}{(R\kappa)^2} [\vec{\dot{\beta}} \vec{R} + c\vec{\beta} \hat{n} - \beta^2 c] \tag{42}$$

Finally the E-field is:

$$\begin{aligned} \vec{E}(t') &= -\vec{\nabla} \Phi(t') - \frac{1}{c} \frac{\partial \vec{A}}{\partial t'} \\ &= \frac{e}{(\kappa R)^2 \kappa} \Big[(\hat{n} - \vec{\beta}) \kappa + \frac{\hat{n}}{c} (\vec{\dot{\beta}} . \vec{R} + c\vec{\beta} . \hat{n} - c\beta^2) \\ & \quad - \frac{\vec{\beta}}{c} (\vec{\dot{\beta}} . \vec{R} + c\vec{\beta} . \hat{n} - c\beta^2) - \frac{e}{cR\kappa^2} \vec{\dot{\beta}} \Big] \end{aligned} \tag{43}$$

After simplification (and using $\overrightarrow{\beta}.\hat{n} = 1 - \kappa$) we end-up with:

$$\begin{aligned}
\overrightarrow{E}(t') &= \frac{e}{\kappa^3 R^2}\left[\frac{\hat{n}}{c}\overrightarrow{\dot{\beta}}.\overrightarrow{R} + (1-\beta^2)\hat{n} - \frac{\overrightarrow{\beta}}{c}\overrightarrow{\dot{\beta}}.\overrightarrow{R} + \overrightarrow{\beta} - \overrightarrow{\beta}\beta^2\right] \\
&= \frac{e}{\kappa^3 R^2}\left[(1-\beta^2)(\hat{n} - \overrightarrow{\beta})\right] + \frac{e}{cR\kappa^3}\left[\overrightarrow{\dot{\beta}}.\hat{n}(\hat{n} - \overrightarrow{\beta}) - \overrightarrow{\dot{\beta}}\kappa\right]
\end{aligned}$$

So finally the $\overrightarrow{E}$ and $\overrightarrow{B}$ fields are given by:

$$\begin{aligned}
\overrightarrow{E}(t') &= \left[\frac{e}{\kappa^3 R^2 \gamma^2}(\hat{n} - \overrightarrow{\beta})\right]_{ret} + \left[\frac{e}{\kappa^3 R}\{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \overrightarrow{\dot{\beta}}]\}\right]_{ret} \\
\overrightarrow{B}(t') &= [n \times \overrightarrow{E}]_{ret}
\end{aligned}$$

where the identity $\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \overrightarrow{\dot{\beta}}] = \overrightarrow{\dot{\beta}}.\hat{n}(\hat{n} - \overrightarrow{\beta}) - \overrightarrow{\dot{\beta}}(1 - \overrightarrow{\beta}.\hat{n})$ was used.

field of a charge moving at constant velocity

$$\overrightarrow{E}(\overrightarrow{x},t) = q\left[\frac{\hat{n}-\overrightarrow{\beta}}{\gamma^2\kappa^3R^2}\right]_{ret} + \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]}{\kappa^3R}\right]_{ret}, \text{ and}$$

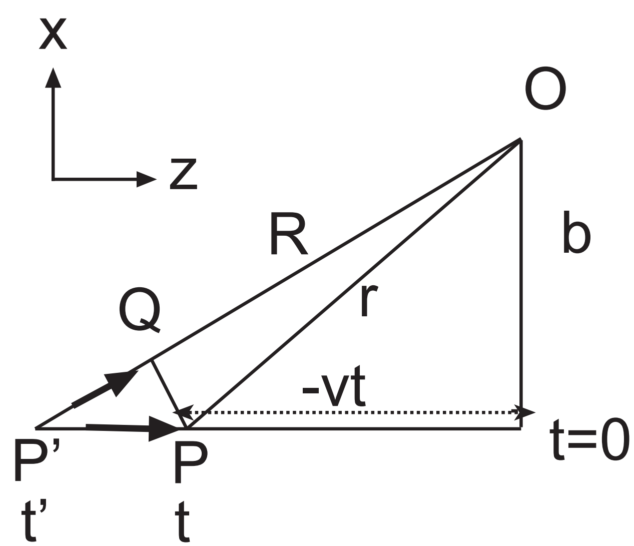
$$\overrightarrow{B}(\overrightarrow{x},t) = [\hat{n}\times\overrightarrow{E}]_{ret}.$$

if $\dot{\beta}=0$, constant velocity then:

$$\overrightarrow{E}(\overrightarrow{x},t) = q\left[\frac{\hat{n}-\overrightarrow{\beta}}{\gamma^2\kappa^3R^2}\right]_{ret}.$$

from part **II**, we know that:

$$\overrightarrow{E}_{\perp}(\overrightarrow{x},t) = \frac{\gamma q b}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}}.$$



$PP' = v(t - t') = \beta R$, $P'Q = PP' \cos(\theta) = \beta R \cos(\theta)$,
$PQ = \beta R \sin(\theta) = \beta R \frac{b}{R} = \beta b$; $QO = R - PP' = (1 - \overrightarrow{\beta} . \widehat{n})R$
$r^2 = QO^2 + PQ^2 = (1 - \overrightarrow{\beta} \widehat{n})^2 R^2 = r^2 - \beta^2 b^2 = (vt)^2 + b^2 - \beta^2 b^2$ So,
$r^2 = \gamma^{-2}[b^2 + \gamma^2 v^2 t^2] = [\kappa^2 R^2]_{ret}$
and $\widehat{x}.(\widehat{n} - \overrightarrow{\beta})_{ret} = \sin(\theta) = \frac{b}{R}$ so that

$$E_x = q \left[\frac{\widehat{x}.(\widehat{n} - \overrightarrow{\beta})}{\gamma^2 \kappa^3 R^2} \right]_{ret} = q \frac{b\gamma}{[b^2 + \gamma^2 v^2 t^2]^{3/2}}. \tag{44}$$

Power radiated

$\overrightarrow{S}(\overrightarrow{x},t).\widehat{n}$: power crossing a unit area, at time t, of a surface that incircles the radiating particle. $\widehat{n}$ is the normal to unit area.
The total energy radiated through the unit area is:

$$W = \int_{-\infty}^{+\infty} dt' \frac{dt}{dt'} \overrightarrow{S}(\overrightarrow{x},t).\widehat{n} = \int_{-\infty}^{+\infty} dt' [\kappa \overrightarrow{S}.\widehat{n}]_{ret}$$

So,

$$\frac{dW}{dt} = [\kappa \overrightarrow{S}.\widehat{n}]_{ret}$$

This is $dP(t')/dA$ or $1/R^2 dP(t')/d\Omega$ so the instantaneous power radiated at time t' per unit solid angle $d\Omega$ is given by:

$$\frac{dP(t')}{d\Omega} = [\kappa \overrightarrow{S}.\widehat{n} R^2]_{ret}$$

From now on, consider only radiation field i.e. R large – this is the "far field" approximation, then

$$\begin{aligned} \overrightarrow{S}.\hat{n} &= \frac{c}{4\pi}[\overrightarrow{E} \times (\hat{n} \times \overrightarrow{E})].\hat{n} \\ &= \frac{c}{4\pi}[E^2 - (\hat{n}.\overrightarrow{E})^2]. \end{aligned}$$

Consider $\hat{n}.\overrightarrow{E}$:

$$\begin{aligned} \hat{n}.\overrightarrow{E} &\propto \hat{n}.\{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \overrightarrow{\dot{\beta}}]\} \\ &\propto \hat{n}.\{(\hat{n}.\overrightarrow{\dot{\beta}})(\hat{n} - \overrightarrow{\beta}) - [\hat{n}.(\hat{n} - \overrightarrow{\beta})]\overrightarrow{\dot{\beta}}\} \\ &\propto \hat{n}.\{(\hat{n}.\overrightarrow{\dot{\beta}})(\hat{n} - \overrightarrow{\beta}) - (1 - \overrightarrow{\beta}.\hat{n})\overrightarrow{\dot{\beta}}\} \\ &= 0. \end{aligned} \tag{45}$$

Hence,

$$\overrightarrow{S}.\hat{n} = \frac{c}{4\pi}E^2 = \frac{q^2}{4\pi c}\left[\frac{|\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \overrightarrow{\dot{\beta}}]|^2}{\kappa^6 R^2}\right]_{ret}$$

$$\frac{dP(t')}{d\Omega} = [\kappa R^2 \overrightarrow{S}.\hat{n}]_{ret} = \frac{q^2}{4\pi c}\left[\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]|^2}{\kappa^5}\right]_{ret}.$$

This is the power radiated per unit solid angle in terms of the charge proper time t'.

If we wants to know $dP(t)/dt$ the power radiated per unit solid angle at the time t it arrives at the enveloping surface, then one must trace back to the associated t' time (retardation).

Note also that $dt = dt'\kappa_{ret} \rightarrow$ if a particle is suddenly (dirac-like) accelerated for a time $\Delta t' = \tau$, a pulse radiation will appear at the observer at time $t = r/c$ and the pulse duration will be $\Delta t = \kappa_{ret}\tau$.

Energy is conserved: total energy radiated=total energy lost by the particle

BUT $\tau\frac{dP(t')}{d\Omega} = \tau\kappa_{ret}\frac{dP(t)}{d\Omega}$; Energy radiated by unit of time $= \kappa_{ret}$ times the energy lost to far-field per unit time.

Instantaneous rate of radiation

$$P(t') = \frac{q^2}{4\pi c}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \,\sin(\theta)\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]|^2}{\kappa^5}$$

$$\begin{aligned}
|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]|^2 &= |(\hat{n}.\overrightarrow{\dot{\beta}})(\hat{n}-\overrightarrow{\beta}) - (1-\overrightarrow{\beta}.\hat{n})\overrightarrow{\dot{\beta}}|^2 \\
&= |(\hat{n}.\overrightarrow{\dot{\beta}})(\hat{n}-\overrightarrow{\beta}) - \kappa\overrightarrow{\dot{\beta}}|^2 \\
&= (\hat{n}.\overrightarrow{\dot{\beta}})^2[1-2\hat{n}.\overrightarrow{\beta}+\beta^2] - 2\kappa\dot{\beta}(\hat{n}.\overrightarrow{\dot{\beta}})(\hat{n}-\overrightarrow{\beta}) + \kappa^2\dot{\beta}^2 \\
&= (\hat{n}.\overrightarrow{\dot{\beta}})^2[1-2\hat{n}.\overrightarrow{\beta}+\beta^2] - 2\kappa(\hat{n}.\overrightarrow{\dot{\beta}})(\hat{n}.\overrightarrow{\dot{\beta}} - \overrightarrow{\beta}.\dot{\beta}) + \kappa^2\dot{\beta}^2
\end{aligned}$$

Using $\overrightarrow{\beta}.\hat{n} = 1-\kappa$ we get:

$$|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]|^2 = -\gamma^{-2}(\hat{n}.\overrightarrow{\dot{\beta}})^2 + 2\kappa(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})(\hat{n}.\overrightarrow{\dot{\beta}}) + \kappa^2\dot{\beta}^2. \quad (46)$$

So,

$$P(t') = \frac{q^2}{4\pi c} 2\pi \int_0^\pi d\theta \sin(\theta) \frac{1}{\kappa^5} [\kappa^2 \dot{\beta}^2 + 2\kappa(\overrightarrow{\dot{\beta}}.\hat{n})(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}}) - \frac{1}{\gamma^2}(\overrightarrow{\dot{\beta}}.\hat{n})^2]$$

$$= \frac{q^2}{2c}\int_0^\pi d\theta \sin\theta \left[\frac{\dot{\beta}^2}{\kappa^3} + \frac{2(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})\dot{\beta}^i n_i}{\kappa^4} - \frac{1}{\gamma^2}\frac{\dot{\beta}^i\dot{\beta}^j n_i n_j}{\kappa^5}\right]. \qquad (47)$$

Recall $\kappa \equiv 1 - \overrightarrow{\beta}.\hat{n} = 1 - \cos\theta$, and let

$$I \equiv \int_0^\pi \frac{\sin\theta d\theta}{(1-\overrightarrow{\beta}.\hat{n})^3} = \int_{-1}^{1} -\frac{du}{(1-\beta u)^3} = \frac{2}{(1-\beta^2)^2} = 2\gamma^4$$

$$J_i \equiv \int_0^\pi \frac{n_i \sin\theta d\theta}{(1-\overrightarrow{\beta}.\hat{n})^4} = \frac{1}{3}\frac{\partial I}{\partial \beta^i} = \frac{8}{3}\frac{\beta_i}{(1-\beta^2)^3} = \frac{8}{3}\beta_i\gamma^6,$$

$$K_{ij} \equiv \int_0^\pi \frac{n_i n_j \sin\theta d\theta}{(1-\overrightarrow{\beta}.\hat{n})^5} = \frac{1}{4}\frac{\partial J_i}{\partial \beta^j} = \frac{2}{3}\frac{\delta_{ij} + \frac{6\beta_i\beta_j}{1-\beta^2}}{(1-\beta^2)^3} = \frac{2}{3}\gamma^6[\delta_{ij} + 6\gamma^2\beta_i\beta_j].$$

$$\Rightarrow P(t') = \tfrac{q^2}{2c}[\dot{\beta}^2 I + 2(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})\dot{\beta}^i J_i - \tfrac{1}{\gamma^2}\dot{\beta}^i\dot{\beta}^j K_{ij}].$$

explicit I, J_i, and K_{ij}:

$$
\begin{aligned}
P(t') &= \frac{q^2}{2c}\left[2\gamma^2\dot\beta^2 + \frac{16}{3}\gamma^6\beta^i\dot\beta_i(\overrightarrow{\beta}\,.\,\overrightarrow{\dot\beta}\,) - \frac{2}{3}\gamma^4(\delta_{ij} + 6\gamma^2\beta_i\beta_j)\dot\beta^i\dot\beta^j\right] \\
&= \frac{q^2}{2c}\left[2\gamma^4\dot\beta^2 + \frac{16}{3}\gamma^6(\overrightarrow{\beta}\,.\,\overrightarrow{\dot\beta}\,)^2 - \frac{2}{3}\gamma^4[\dot\beta^2 + 6\gamma^2(\overrightarrow{\beta}\,.\,\overrightarrow{\dot\beta}\,)^2\right] \\
&= \frac{2q^2}{3c}\left[\gamma^4\dot\beta^2 + \gamma^6(\overrightarrow{\beta}\,.\,\overrightarrow{\dot\beta}\,)^2\right] \\
&= \frac{2q^2}{3c}\gamma^6\left[(1-\beta^2)\dot\beta^2 + (\overrightarrow{\beta}\,.\,\overrightarrow{\dot\beta}\,)^2\right] = \frac{2q}{3c}\gamma^6[\dot\beta^2 - \dot\beta^2\beta^2(1-\cos^2\Phi) \\
&= \frac{2q^2}{3c}\gamma^6[1 - \dot\beta^2\beta^2\sin^2\Phi] = \frac{2q^2}{3c}\gamma^6[\dot\beta^2 - (\overrightarrow{\beta}\times\overrightarrow{\dot\beta}\,)^2] \qquad (48)
\end{aligned}
$$

This is the relativistic generalization of the Larmor's results (to recover the standard Larmor power consider $\beta \to 0$).

example 1: radiative energy loss from a linear accelerator

In linear accelerator (or “linac”), $\overrightarrow{\dot{\beta}} \parallel \overrightarrow{\beta}$. In order to calculate $P(t')$, we need to evaluate $\dot{\beta}$. From $p = \gamma\beta mc$ we have:

$$\begin{aligned} \dot{p} &= mc(\dot{\gamma}\beta + \gamma\dot{\beta}) = mc[(\gamma^3\beta\dot{\beta})\beta + \gamma\dot{\beta}] \\ &= \gamma mc\left(\frac{\beta^2}{1-\beta^2} + 1\right)\dot{\beta} = \gamma^3 mc\dot{\beta}. \end{aligned} \tag{49}$$

So

$$P(t') = \frac{2}{3}\frac{q^2}{m^2c^3}\dot{p}^2 \text{ [JDJ Eq. (14.27)]}$$

Since $P \propto m^{-2}$ lighter particle are subject to higher losses. The rate of momentum change is proportional to the particle energy change:$\dot{p} = dE/dz$ (consider particle being accelerated along the $\hat{z}$-direction).

The question is for what energy gain does radiative effects start to influence the dynamics. Let $P_{ext} \equiv [dE/dt]_{ret}$ be the power associated to the external (accelerating force) then the radiative effect are comparable to external force effects when:

$$\frac{P_{rad}}{P_{ext}} = \frac{P(t')}{vdE/dz} = \frac{2}{3}\frac{q^2}{m^2c^3}\left[\frac{1}{v}\frac{dE}{dz}\right]_{ret} \sim 1.$$

Consider e-: typically $v \simeq c$, and $q = e$ then

$$\frac{P_{rad}}{P_{ext}} = \frac{2}{3}\frac{e^2/(mc^2)}{mc^2}\left[\frac{dE}{dz}\right]_{ret}$$

So $P_{rad} \simeq P_{ext}$ if $dE/dz \simeq mc^2/r_e = 0.511/(2.8 \times 10^{-15}) = 2 \times 10^{14}$ MeV/m

compare to 100 MeV/m state-of-art conventional accelerator or to 30 Gev/m plasma-based accelerator *; we see that radiative effects have negligible impact on the dynamics of e- beams.

*W. Leeman, *et al.*, *Nature Phys.* **2**, 696-699 (October 2006), also *The Economist*, September 28th, 2006 example 2: radiative energy loss in a circular accelerator

In circular accelerator acceleration is centripetal: $\overrightarrow{\dot{\beta}} \perp \overrightarrow{\beta}$ so

$$\dot{\beta}^2 - (\overrightarrow{\beta} \times \overrightarrow{\dot{\beta}})^2 = \dot{\beta}^2(1 - \beta^2) = \frac{\dot{\beta}^2}{\gamma^2}$$

So the radiated power is:

$$P(t') = \frac{2}{3}\frac{q^2 c}{R^2}(\beta\gamma)^4 = \frac{2}{3}\frac{q^2 c}{R^2}\beta^4 \left[\frac{E}{mc^2}\right]^4,$$

where E is the total energy. The revolution period is $T = 2\pi R/(\beta c)$, and $P = \frac{\Delta E}{T}$. So the radiative loss per turn is:

$$\Delta E = PT = \frac{2}{3}\frac{q^2 c}{R^2}\beta^4 \left[\frac{E}{mc^2}\right]^4 \frac{2\pi R}{\beta c}$$

that is:

$$\Delta E = \frac{4\pi}{3}\frac{q^2}{R}\beta^3 \left[\frac{E}{mc^2}\right]^4 \text{ [JDJ Eq. (14.32)]}$$

Consider an e- synchrotron accelerator, the energy loss per turn and per electon is:

$$\Delta E \simeq \frac{4\pi}{3}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4 .$$

Take $E = 1$ TeV, $R = 2$ km we then have:

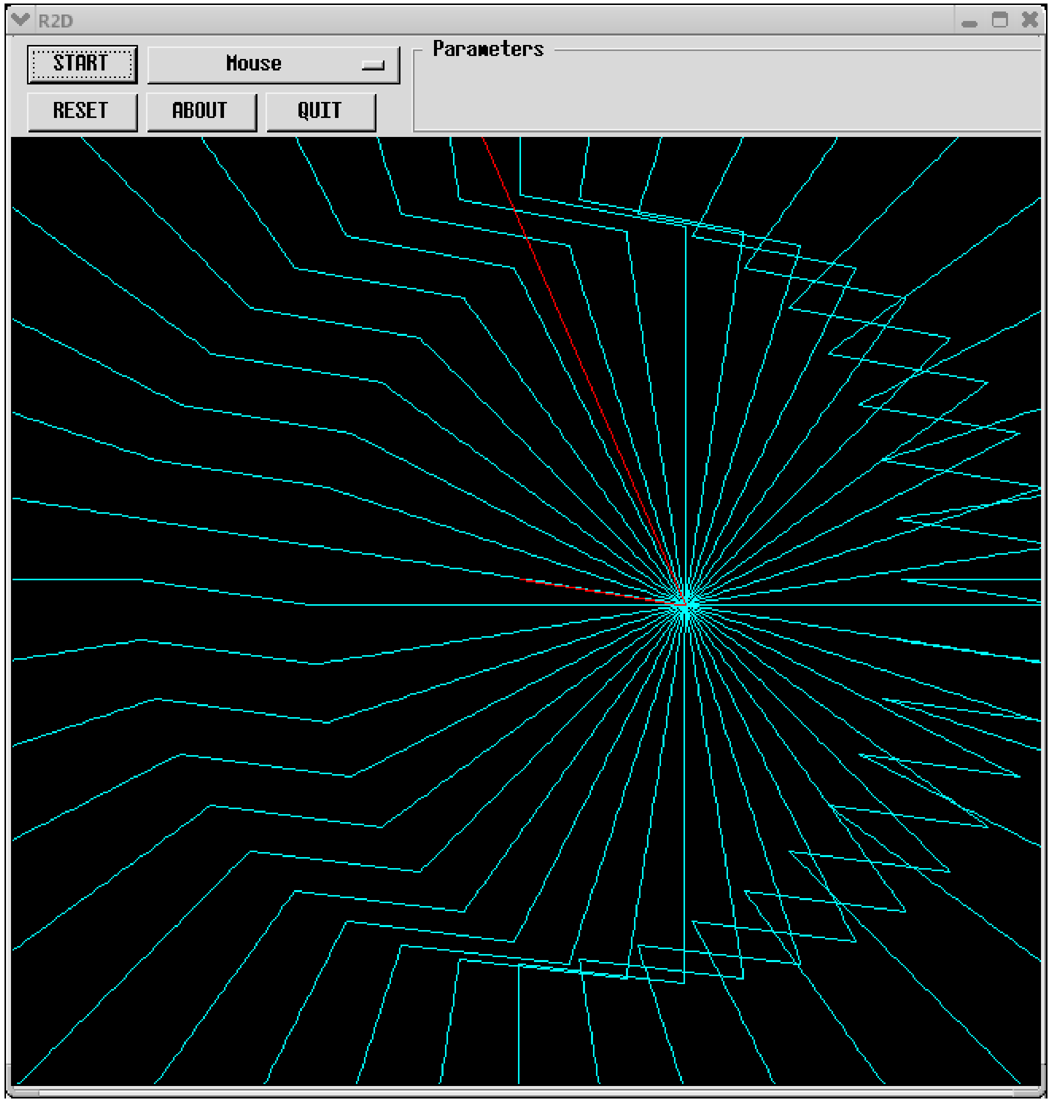
$$\Delta E \text{ [eV] } = \frac{1}{3\epsilon_0}\frac{e}{R}\left(\frac{E}{mc^2}\right)^4 = 44.2 \text{ TeV !!}$$

For protons however we gain a factor $(m_e/m_p)^4 = 1/1836^4$ so

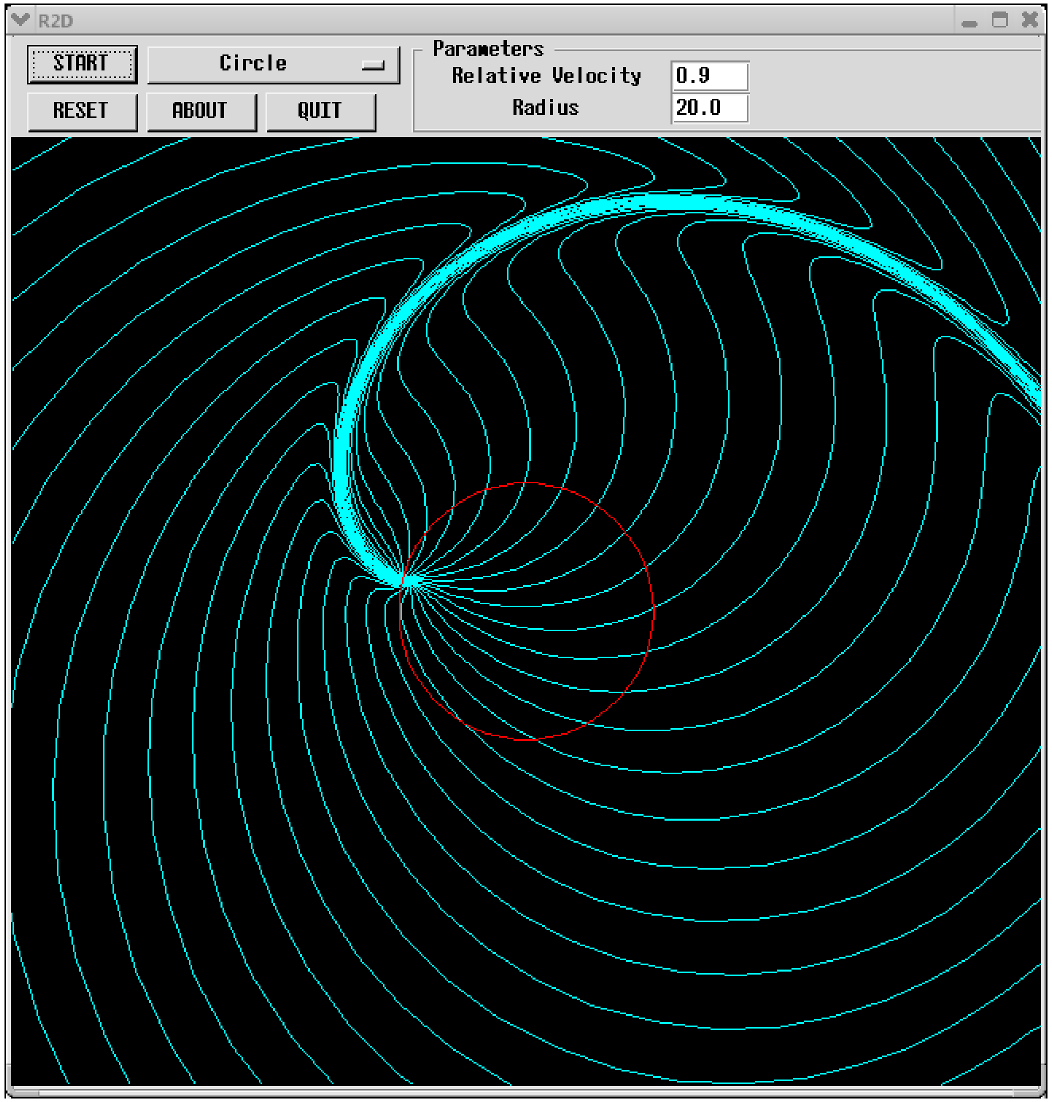
$$\Delta E_{proton} \simeq 4 \text{ eV}$$

High energy physics circular accelerator use proton (or ions) reasons for Tevatron at FNAL or LHC at CERN. One can however use e-/e+ storage ring as a copious source of radiation for use or for "cooling" =radiation damping in the internation linear collider proposal.

Field line associated to a linearly moving charge



Field line associated to a moving charge in circular motion



Angular Distribution of radiation emitted by an accelerated charge

$$\begin{aligned}
\frac{dP(t')}{d\Omega} &= \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]|^2}{\kappa^5} \\
&= \frac{q^2}{4\pi c}\frac{\kappa^2\dot{\beta}^2+2\kappa(\overrightarrow{\dot{\beta}}.\hat{n})(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})-\gamma^{-2}(\overrightarrow{\dot{\beta}}.\hat{n})^2}{\kappa^5} \qquad (50)
\end{aligned}$$

where we have used Eq.46.

<u>Case of linear motion</u>

$\overrightarrow{\beta}.\hat{n}=\beta\cos\theta$, $\overrightarrow{\dot{\beta}}.\hat{n}=\dot{\beta}\cos\theta$, $\kappa=1-\overrightarrow{\beta}.\hat{n}=1-\beta\cos\theta$, and numerator of $dP(t')/d\Omega$ is:

$$\begin{aligned}
& \dot{\beta}^2[\kappa^2+2\kappa\beta\cos\theta-(1-\beta^2)\cos^2\theta] \\
= & \dot{\beta}^2[(\kappa^2+2\kappa\beta\cos\theta+\beta^2\cos^2\theta)-\cos^2\theta] \\
= & \dot{\beta}^2[(\kappa+\beta\cos\theta)^2-\cos^2\theta]=\dot{\beta}^2\sin^2\theta. \qquad (51)
\end{aligned}$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \text{ [JDJ, Eq.(14.39)]} \tag{52}$$

The location of peak intensity are given by:

$$\begin{aligned} 0 &= \frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right) \\ &= \frac{\sin(\theta)\left(2\cos(\theta)+3\beta(\cos(\theta))^2-5\beta\right)}{(1-\beta\cos\theta)^4} \end{aligned} \tag{53}$$

whose solutions are:

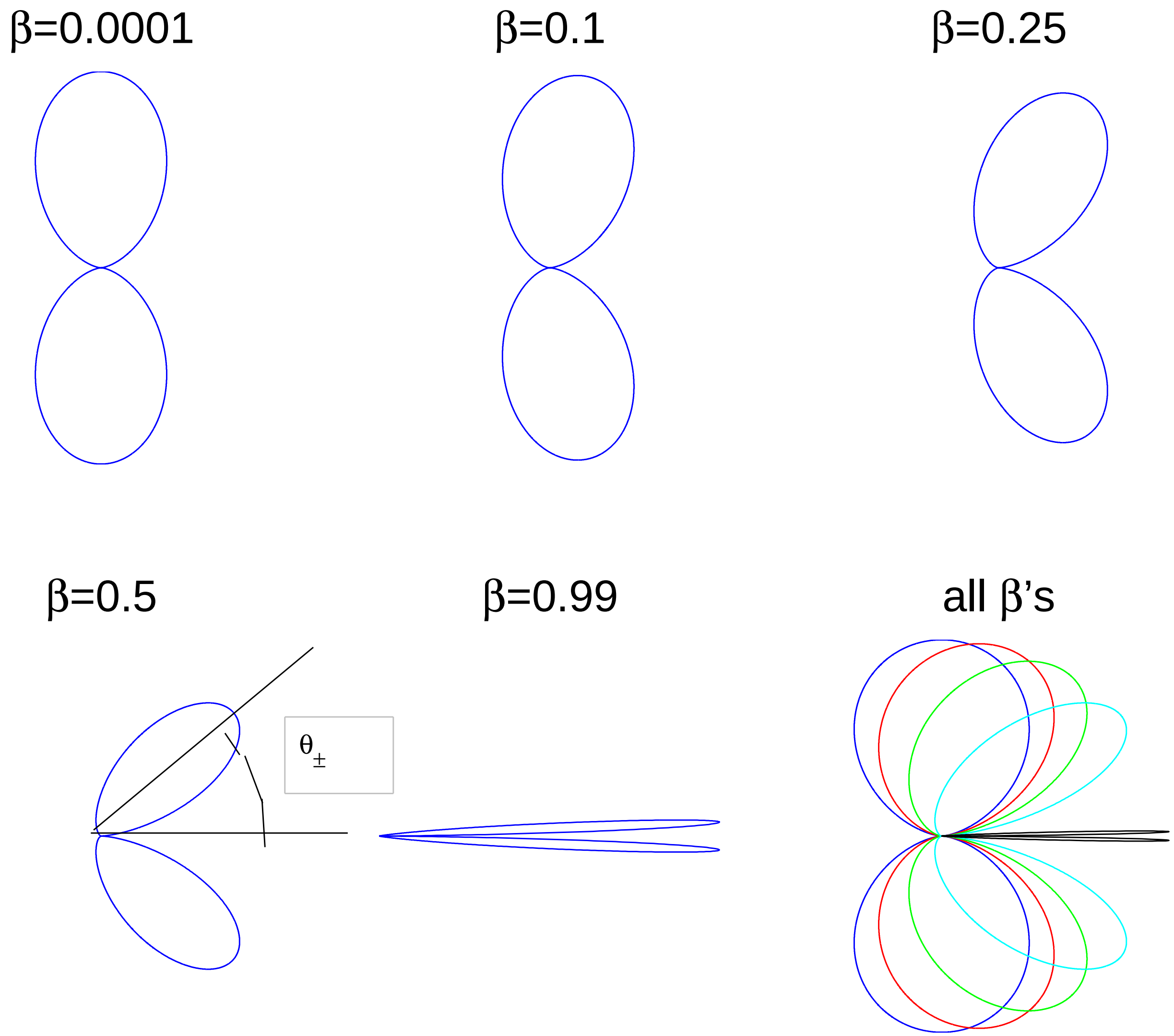
$$[\cos\theta]_\pm = \frac{1}{3\beta}[-1 \pm (1+15\beta^2)^{1/2}] \tag{54}$$

Only $[\cos\theta]_+$ is viable since we must have $|\cos(\theta)| < 1$. So finally

$$\theta_\pm = \pm\arccos\left[\frac{1}{3\beta}[-1+(1+15\beta^2)^{1/2}]\right] \xrightarrow{\beta\to 1} \pm\frac{1}{2\gamma} \tag{55}$$

these are locations of maximum in power.

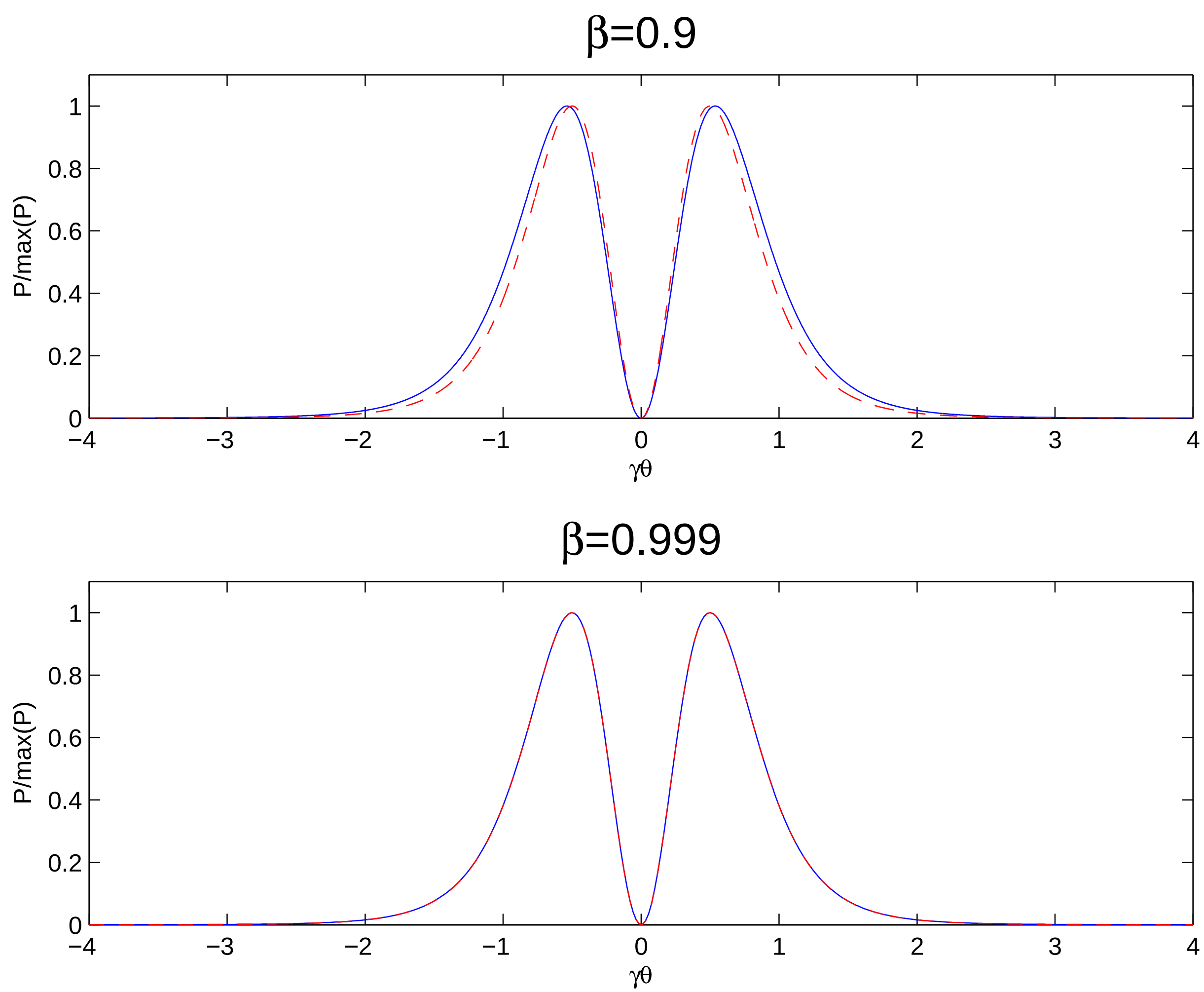
Angular distribution for the case of linear motion



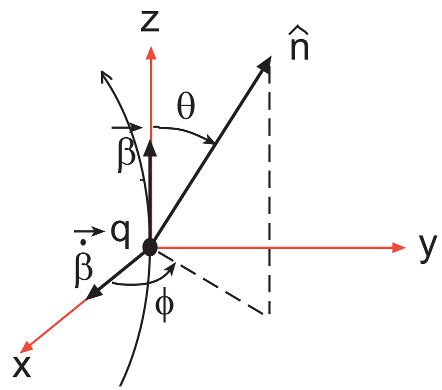
Ultra-relativistic limit: as the $\beta \longrightarrow 1$ the intensity angular distribution is contained within small angle (so $\theta \ll 1$). The power angular distribution then becomes:

$$\frac{dP(t')}{d\Omega} \simeq \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{\theta^2}{(1-\beta(1-\frac{\theta^2}{2}))^5} = \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{32\theta^2}{2(1-\beta)+\beta\theta^2))^5}$$
$$\simeq \frac{8}{\pi}\frac{\dot{\beta}^2}{c^2}\frac{\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5} \text{ [JDJ, Eq.(14.41)].} \qquad (56)$$

Comparison of exact θ-dependence (solid line) with ultra-relativistics approximation (dash line) for two cases of β:



Case of circular motion



snapshot of motion taken at time t'.

$$\hat{z} = \cos\theta\hat{n} - \sin\theta\hat{\theta}$$
$$\hat{x} = \sin\theta\cos\phi\hat{n} + \cos\theta\sin\phi\hat{\theta} - \sin\phi\hat{\phi}$$

Thus $\vec{\beta}.\hat{n} = \beta\cos\theta$, $\vec{\beta}.\vec{\dot{\beta}} = 0$, and $\vec{\dot{\beta}}.\hat{n} = \dot{\beta}\sin\theta\cos\phi$

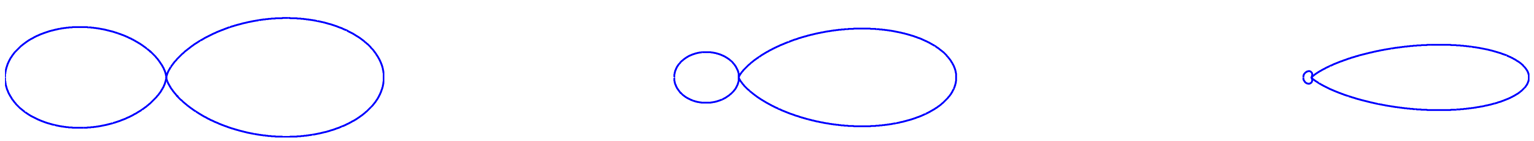
$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^2}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \quad \text{[JDJ Eq.(14.44)]} \tag{57}$$

Unlike linear motion, the power angular distribution peaks at $\theta = 0$.

Considering the ultra-relativistic limit ($\beta \to 1$, $\theta \ll 1$).

$$\frac{dP(t')}{d\Omega} = \frac{8}{\pi}\frac{q^2}{c^2}\frac{\dot{\beta}^2}{(1+\gamma^2\theta^2)^3}\gamma^6\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] \quad \text{[JDJ Eq.(14.44)]} \tag{58}$$

β=0.05 β=0.2 β=0.5



(distribution evaluated in the plan $\phi = 0$)

A part from a different in the intensity distributions for linear and circular motion, there is also a difference in total radiated power:

$$
\begin{aligned}
P_{Linear} &= \frac{2}{3} q^2 m^2 c^3 \dot{p}^2 \\
P_{Circular} &= \frac{2}{3} q^2 c \gamma^4 \dot{\beta}^2 = \frac{2}{3} q^2 m^2 c^3 \gamma^2 \dot{p}^2
\end{aligned}
$$

Thus

$$\frac{P_{Circular}}{P_{Linear}} = \gamma^2$$

For a given applied force, there is γ^2 times more radiation energy if the force is applied perpendicular to the charge's velocity that is applied parallel to the velocity.

Radiation Spectrum

Go in the observer's frame:

$$\begin{aligned}\frac{dP(t)}{d\Omega} &= \frac{1}{\kappa(t')}\frac{dP(t')}{d\Omega} \\ &= \frac{q^2}{4\pi c}\left[\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2}{\kappa^6}\right]_{ret} \equiv |\overrightarrow{A}(t)|^2 \qquad (59)\end{aligned}$$

wherein

$$\overrightarrow{A}(t) = \sqrt{\frac{c}{4\pi}}[R\overrightarrow{E}]_{ret} \qquad (60)$$

to obtain the power spectrum of the radiation we need to work in the frequency domain, so decompose $\overrightarrow{A}$ as:

$$\overrightarrow{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\,\overrightarrow{A}(t)e^{i\omega t}, \qquad (61)$$

and reciprocally:

$$\overrightarrow{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \overrightarrow{A}(\omega) e^{-i\omega t}, \tag{62}$$

From Parseval's theorem the total energy radiated per $d\Omega$ is

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} dt |\overrightarrow{A}(t)|^2 = \int_{-\infty}^{+\infty} d\omega |\overrightarrow{A}(\omega)|^2 \tag{63}$$

If $\overrightarrow{A}(t) \in \mathbb{R}$, then $\overrightarrow{A}^*(\omega) = \overrightarrow{A}(\omega)$ and:

$$\frac{dW}{d\Omega} = 2 \int_{0}^{\infty} d\omega |\overrightarrow{A}(\omega)|^2 \tag{64}$$

So the radiation spectrum per unit of solid angle is:

$$\frac{d^2 I(\widehat{n}, \omega)}{d\Omega d\omega} = 2|A(\omega)|^2 \tag{65}$$

Thus we need to evaluate $\overrightarrow{A}(\omega)$

$$\overrightarrow{A}(t) = \frac{q}{\sqrt{4\pi c}} \left[\frac{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]}{\kappa^3} \right]_{ret} \tag{66}$$

and so,

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}} \int_{-\infty}^{+\infty} dt \left[\frac{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]}{\kappa^3} \right]_{ret} e^{i\omega t} \tag{67}$$

since the quantity [...] must be evaluated at the retarded time, let $dt = \kappa(t')dt'$ and $t = t' + \frac{R(t')}{c}$ then the integral becomes:

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}} \int_{-\infty}^{+\infty} dt' \frac{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]}{\kappa^2} e^{i\omega(t' + \frac{R(t')}{c})} \tag{68}$$

In the far-field regime (large $|\overrightarrow{x}|$) we have: $\hat{n} = \frac{\overrightarrow{x} - \overrightarrow{r}(t')}{|\overrightarrow{x} - \overrightarrow{r}(t')|} \simeq \hat{x}$ constant in time. And $R = x - \overrightarrow{r}.\hat{n} + \mathcal{O}(1/x)$.

In the far-field regime the argument of the exponential rewrites:

$$\Xi = i\omega[t' + \frac{R(t)}{c}] = i\omega x + i\omega[t' - \frac{\hat{n}.\overrightarrow{r}(t')}{c}] \tag{69}$$

we henceforth ignore the term $i\omega x$ since it has no contribution (the final result is $\propto |A(\omega|^2)$ and define

$$\Xi(t') = i\omega[t' - \frac{\hat{n}.\overrightarrow{r}(t')}{c}], \tag{70}$$

we have

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty} dt \frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]}{\kappa^2} e^{\Xi(t)}, \tag{71}$$

and the intensity distribution takes the form

$$\frac{d^2 I(\hat{n},\omega)}{d\Omega d\omega} = 2A^2(\omega) = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt \frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]}{\kappa^2} e^{\Xi(t)}\right|^2. \tag{72}$$

To follow JDJ, let's show that the vectorial quantity in the integral can be written as a time-derivative, in the far-field approximation. Consider

$$\frac{\hat{n} \times (\hat{n} \times \overrightarrow{\dot{\beta}})}{\kappa}, \tag{73}$$

and let's compute

$$\frac{d}{dt}\left[\frac{\hat{n} \times (\hat{n} \times \overrightarrow{\beta})}{\kappa}\right] = \frac{(-\dot{\kappa}\hat{n} + (1-\kappa)\dot{\hat{n}} - \overrightarrow{\dot{\beta}})\kappa - \dot{\kappa}[(1-\kappa)\hat{n} - \overrightarrow{\beta}]}{\kappa^2} \tag{74}$$

It is straightforward (see Eq. 37) to show that $\dot{\hat{n}} \propto 1/R$ and $\dot{\kappa} = -\overrightarrow{\dot{\beta}}.\hat{n} - \overrightarrow{\beta}.\dot{\hat{n}} = -\overrightarrow{\dot{\beta}}.\hat{n} + \mathcal{O}(1/R)$. So

$$\begin{aligned} \frac{d}{dt}[...] &= \frac{1}{\kappa^2}\left\{[(\overrightarrow{\dot{\beta}}.\hat{n})\hat{n} - 0 - \overrightarrow{\dot{\beta}}]\kappa + (\overrightarrow{\dot{\beta}}.\hat{n})[(1-\kappa)\hat{n} - \overrightarrow{\beta}]\right\} \\ &= \frac{1}{\kappa^2}\left\{-\overrightarrow{\dot{\beta}}\kappa + (\overrightarrow{\dot{\beta}}.\hat{n})(\hat{n} - \overrightarrow{\beta})\right\} + \mathcal{O}(1/R) = \frac{1}{\kappa^2}\left\{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \overrightarrow{\dot{\beta}}]\right\}. \end{aligned}$$

So the vectorial quantity is a time-derivative and we can write:

$$\begin{aligned}\overrightarrow{A}(\omega) &= \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty} dt\frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\overrightarrow{\beta})}{\kappa}\right]e^{\Xi(t)} \\ &= \frac{q}{2\pi\sqrt{2c}}\left\{\left|\left[\frac{\hat{n}\times(\hat{n}\times\overrightarrow{\beta})}{\kappa}\right]e^{\Xi(t)}\right|_{-\infty}^{+\infty} - i\omega\int_{-\infty}^{+\infty} dt\left[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})\right]e^{\Xi(t)}\right\}\end{aligned}$$

The first integral is zero (in principle on should introduce a decay term $e^{-\epsilon|t|}$, with $\epsilon > 0$, perform the integral and take the limit $\epsilon \rightarrow 0$).We finally have:

$$\frac{d^2I(\hat{n},\omega)}{d\Omega d\omega} = \frac{q^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{+\infty} dt[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})]e^{i\omega[t'-\frac{\hat{n}.\overrightarrow{r}(t)}{c}]}\right|^2 \tag{75}$$

Nota: $[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})] = \beta\sin\theta = |\hat{n}\times\overrightarrow{\beta}|$ where $\theta = \angle(\hat{n},\overrightarrow{\beta})$.

Case of circular motion

$$
\begin{aligned}
\widehat{n} &= \sin\theta\widehat{y} + \cos\theta\widehat{z}, & (76)\\
\overrightarrow{\beta} &= \beta[\sin(\omega_0 t')\widehat{x} + \cos(\omega_0 t')\widehat{z}], & (77)\\
\widehat{\epsilon}_{\|} &= \widehat{x}, & (78)\\
\widehat{\epsilon}_{\perp} &= \widehat{n}\times\widehat{x} = -\sin\theta\widehat{z} + \cos\theta\widehat{y}. & (79)
\end{aligned}
$$

$$\begin{aligned}\hat{n}\times(\hat{n}\times\overrightarrow{\beta}) &= (\hat{n}.\overrightarrow{\beta})\hat{n}-\overrightarrow{\beta} \\ &= \beta[c_{\omega_0 t}c_\theta\hat{y}+c_{\omega_0 t}(c_\theta^2-1)\hat{z}-c_{\omega_0 t}\hat{x} \\ &= \beta[-s_{\omega_0 t}\hat{\epsilon}_{\|}+c_{\omega_0 t}s_\theta\hat{\epsilon}_{\perp}] \end{aligned} \tag{80}$$

Let's now consider the argument of the exponential function Ξ. First we note that $\hat{n}.\overrightarrow{r}=r\cos\theta\cos(\pi/2-\omega_0 t')=r\sin(\omega_0 t')\cos\theta$ and

$$\Xi=i\omega(t'-\frac{\hat{n}.\overrightarrow{r}}{c})=\omega[t'-\frac{r}{c}\sin(\omega_0 t')\cos\theta] \tag{81}$$

Also if P catch an impulse of radiation from q: q's radiation is confined in forward direction, θ is small, and the pulse originated near $\omega_0 t\simeq 0$. Under these approximations:

$$\lim_{\theta\ll 1,\omega_0 t\ll 1}\hat{n}\times(\hat{n}\times\overrightarrow{\beta})=\beta(-\omega_0 t\hat{\epsilon}_{\|}+\theta\hat{\epsilon}_{\perp}) \tag{82}$$

and,

$$\begin{aligned}
\lim_{\theta\ll 1,\omega_0 t\ll 1} \frac{1}{i}\Xi &= \omega\left\{t' - \frac{r}{c}[\omega_0 t' - \frac{1}{6}(\omega_0 t')^3](1-\frac{\theta^2}{2})\right\} \\
&= \omega\left\{(1-\beta)t' + \frac{\beta t'}{2}\theta^2 + \frac{1}{6}\frac{r}{c}(\omega_0 t')^3\right\} \\
&= \frac{\omega t'}{2}(\gamma^{-2} + \beta\theta^2) + \frac{\omega\beta}{6\omega_0}(\omega_0 t')^3. \qquad (83)
\end{aligned}$$

The spectral energy density is:

$$\begin{aligned}
\frac{d^2 I}{d\Omega d\omega} &= \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt\beta(-\omega_0 t\hat{\epsilon}_{\parallel} + \theta\hat{\epsilon}_{\perp})e^{\Xi}\right|^2 \\
&= \left| - A_{\parallel}(\omega)\hat{\epsilon}_{\parallel} + A_{\perp}(\omega)\hat{\epsilon}_{\perp}\right|^2 \qquad (84)
\end{aligned}$$

This displays the two polarization associated to the radiation. Nota: $\parallel$ and $\perp$ polarizations are also respectively refer to as σ and π-polarizations.

where

$$\begin{pmatrix} A_\parallel \\ A_\perp \end{pmatrix} = \frac{q\omega}{2\pi\sqrt{c}} \int_{-\infty}^{+\infty} dt \begin{pmatrix} \omega_0 t \\ \theta \end{pmatrix} e^{i\frac{\omega}{2}[(\gamma^{-2}+\theta^2)t+\frac{1}{3\omega_0}(\omega_0 t')^3]}. \tag{85}$$

let $x = \frac{\omega_0 t}{\sqrt{\gamma^{-2}+\theta^2}}$, $dt = \frac{1}{\omega_0}\sqrt{\gamma^{-2}+\theta^2}dx$; and let $\xi \equiv \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2}$, then

$$\begin{pmatrix} A_\parallel(\omega) \\ A_\perp(\omega) \end{pmatrix} = \frac{q\omega}{2\pi\sqrt{c}} \int_{-\infty}^{+\infty} dx \begin{pmatrix} (\gamma^{-2}+\theta^2)x\frac{1}{\omega_0} \\ (\gamma^{-2}+\theta^2)^{1/2}\theta\frac{1}{\omega_0} \end{pmatrix} e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]}. \tag{86}$$

we have the identity:

$$\int_{-\infty}^{+\infty} dt e^{i(xt+at^3)} = \frac{2\pi}{(2a)^{1/3}} A_i\left(\frac{x}{(3a)^{1/3}}\right),$$

where A_i is the Airy function, Note also that $A_i(x) = \frac{1}{\pi}\sqrt{\frac{1}{3}x}K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$. Thus:

$$\int_{-\infty}^{+\infty} dx e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]} = \frac{2\pi}{(3\xi/2)^{1/3}} A_i\left[\left(\frac{3\xi}{2}\right)^{2/3}\right] = \frac{2}{\sqrt{3}}K_{1/3}(\xi). \tag{87}$$

For the other integral. Note that

$$\int_{-\infty}^{+\infty} dt t e^{i(xt+at^3)} = \frac{1}{i}\frac{d}{dx}\int_{-\infty}^{+\infty} t e^{i(xt+at^3)} dt = \frac{2\pi}{(2a)^{1/3}} A_i'\left(\frac{x}{(3a)^{1/3}}\right),$$

The prime denote the differentiation w.r.t. total argument of A_i. Inserting $a = \xi/2$, and $x = 3\xi/2$ we get:

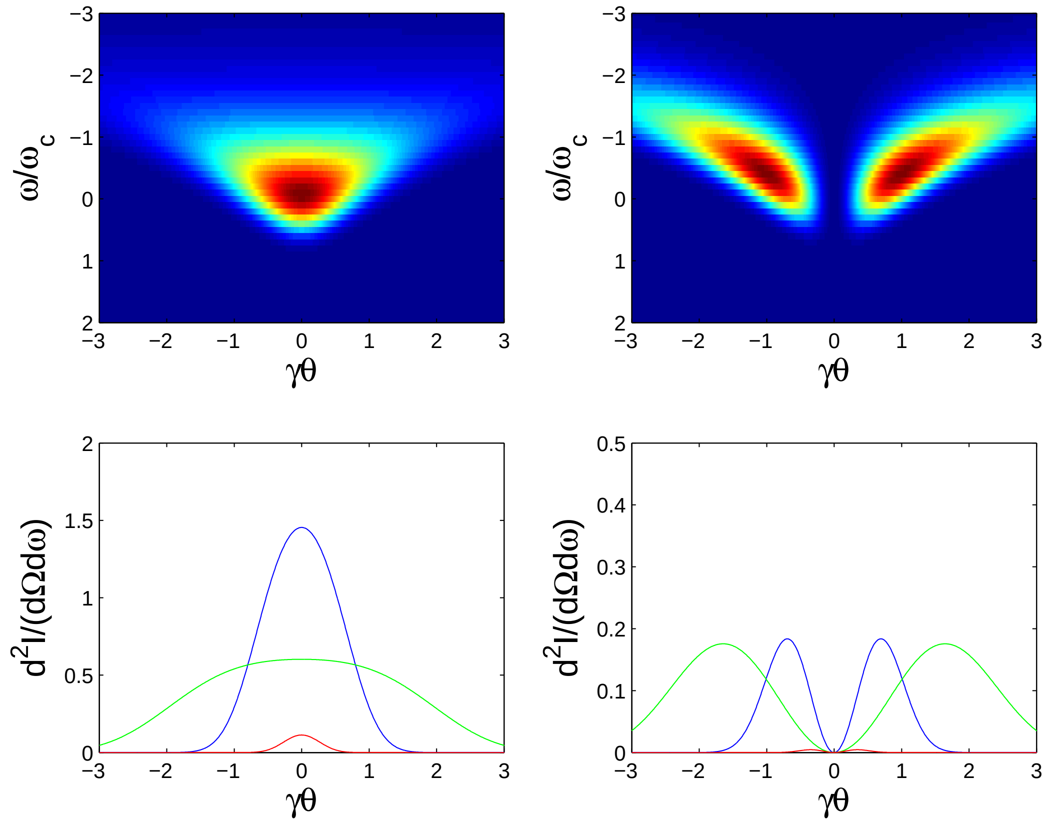
$$\int_{-\infty}^{+\infty} x e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]} dx = \frac{2\pi}{(3\xi/2)^{1/3}} A_i'\left[\left(\frac{3\xi}{2}\right)^{2/3}\right] = -\frac{1}{i}\frac{2}{\sqrt{3}} K_{2/3}(\xi). \quad (88)$$

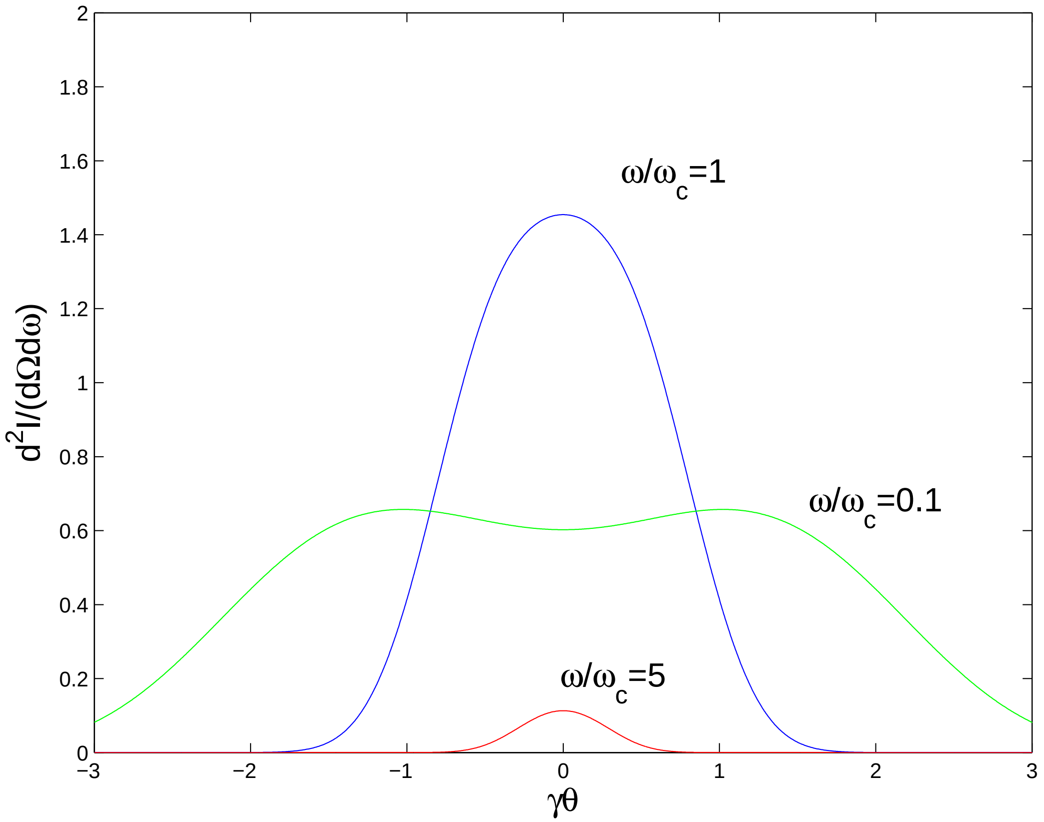
where we have used: $A_i'(x) = \frac{-1}{\pi}\sqrt{\frac{1}{3}x} K_{2/3}\left(\frac{2}{3}x^{3/2}\right)$. So the spectral intensity per unit of solid angle takes the form:

$$\begin{aligned} \frac{d^2 I}{d\Omega d\omega} &= |A_{\|}(\omega)|^2 + |A_{\perp}(\omega)|^2 \\ &= \frac{q^2}{3\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 (\gamma^{-2}+\theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2} K_{1/3}^2(\xi)\right] \end{aligned}$$

or, introducing $\xi = \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2} \equiv \frac{1}{2}\frac{\omega}{\omega_c}[1+\gamma^2\theta^2]^{3/2}$:

$$\frac{d^2 I}{d\Omega d\omega} = \frac{3q^2}{\pi^2 c}\xi^2 \frac{1}{\gamma^{-2}+\theta^2}\left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\xi)\right]$$





High frequency radiation occupies $\theta < \gamma^{-1}$ ($\ll \gamma^{-1}$ for $\omega \gg \omega_c$) and low frequency radiation occupies $\theta > \gamma^{-1}$. It is usual to also define a critical angle as $\theta_c = \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)$

For low frequency $\omega \ll \omega_c$, the frequency spectrum integrated over the solid angle is:

$$\begin{aligned}\frac{dI}{d\omega} &\simeq 2\pi\theta_c \left[\frac{d^2I}{d\omega d\Omega}\right]_{\theta=0} \\ &= \frac{2\pi}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/3}\frac{3}{\pi^2}\frac{q^2}{c}\gamma^2[\xi^2 K^2_{2/3}(\xi)]_{\theta=0}. \end{aligned} \tag{89}$$

$\xi(0) = \frac{\omega}{2\omega_c} \ll 1$ so that

$$[\xi K_{2/3}(\xi)]^2_{\theta=0} \simeq \left[\frac{\Gamma(2/3)}{2^{1/3}}\right]^2 [\xi(0)]^{2/3} \simeq \left(\frac{\omega}{2\omega_c}\right)^{2/3}. \tag{90}$$

So

$$\frac{dI}{d\omega} \simeq \frac{6}{\pi}\frac{q^2}{c}\gamma\left(\frac{\omega}{2\omega_c}\right)^{1/3} = \frac{6}{\pi}\frac{q^2}{c}\gamma\left(\frac{\omega}{3\gamma^3\omega_0}\right)^{1/3} \propto \omega^{1/3} \tag{91}$$

for $\omega \ll \omega_c$, so it is very broad γ-independent spectrum.

Angular distribution:

we need to calculate $\int_0^\infty d\omega \frac{d^2I}{d\omega d\Omega}$. Do the variable change $\xi = \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2}$ then:

$$\begin{aligned}\frac{dI}{d\Omega} &= \frac{3q^2}{\pi^2 c}\frac{3\omega_0}{[\gamma^{-2}+\theta^2]^{5/2}}\int_0^\infty \xi^2\left\{K_{2/3}^2(\xi)+\frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\xi)\right\}d\xi \\ &= \frac{9}{\pi^2}\frac{q^2}{c}\frac{\gamma^5\omega_0}{[1+(\gamma\theta)^2]^{5/2}}\left[\frac{7\pi^2}{144}+\frac{5\pi^2}{144}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)}\right] \end{aligned} \tag{92}$$

where we have used the identity:

$$\int_0^\infty \omega^2 K_\mu^2(a\omega)d\omega = \frac{\pi^2}{32a^3}\frac{1-4\mu^2}{\cos(\pi\mu)}$$

Thus we finally have:

$$\frac{dI}{d\Omega} = \frac{7}{16}\frac{q^2}{c}\frac{\gamma^5\omega_0}{[\gamma^{-2}+\theta^2]^{5/2}}\left[1+\frac{5}{7}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)}\right] \text{ [JDJ, Eq.(14.80)]}$$

The total energy radiated is $\Delta W = \int d\Omega \frac{dI}{d\Omega} = 2\pi \int d\theta \frac{dI}{d\theta}$ the integral on θ should be within $[-\pi, \pi]$ however because we did a small angle approximation and since $dI/d\Omega$ is significant only for $\gamma\theta < 1$ we do this integral from $[-\infty, \infty]$:

$$
\begin{aligned}
\Delta W &= 2\pi \int_0^\infty d\theta \frac{dI}{d\theta} \\
&= \frac{7\pi}{8}\frac{q^2}{c}\gamma^5\omega_0 \int_{-\infty}^{+\infty} \left[\frac{1}{(1+\gamma^2\theta^2)^{5/2}} + \frac{5}{7}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^{7/2}}\right] \\
&= \frac{7\pi}{8}\frac{q^2}{c}\gamma^5\omega_0 \left[\frac{4}{3\gamma} + \frac{4}{15\gamma}\right] = \frac{7\pi}{6}\frac{q^2}{c}\gamma^4\omega_0 \left[1 + \frac{1}{7}\right] \qquad (93)
\end{aligned}
$$

There is 7 times more energy radiated in the $\|$-polarization than in the $\perp$-polarization. The total energy radiated is

$$\Delta W = \frac{4\pi}{3}\frac{q^2}{c}\gamma^4\omega_0$$

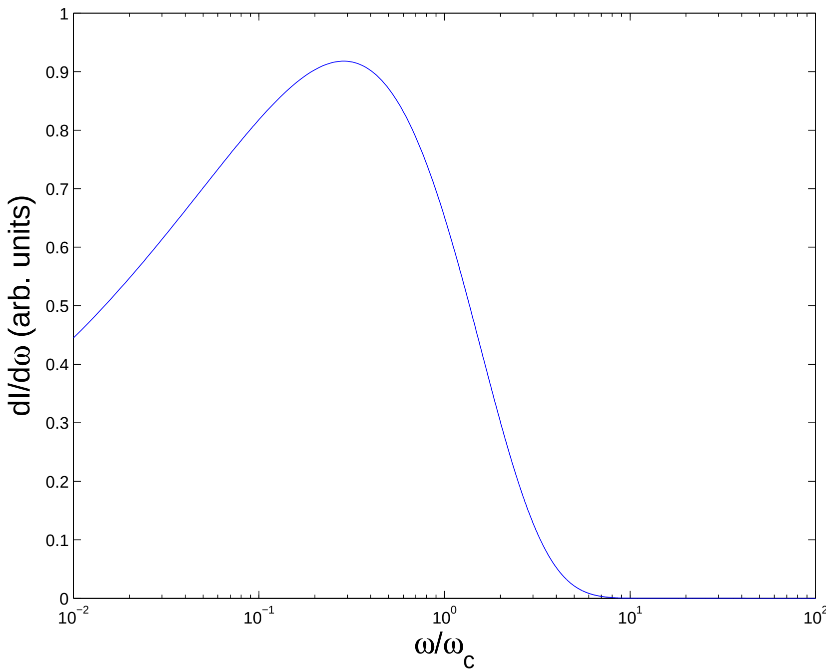
where $\omega_0 = c/r$.

Let show that the previous result is in agreement with the radiated power associated to circular motion we computed earlier in this chapter.

$$\begin{aligned}
\Delta W_{circ} &= P_{circ}\frac{2\pi}{\omega_0} = \left(\frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\dot{p}^2\right)\frac{2\pi}{\omega_0} \\
&= \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2(\gamma m r\omega_0^2)^2\frac{2\pi}{\omega_0} = \frac{4\pi}{3}\frac{q^2}{c^3}\gamma^4 r^2\omega_0^3 \\
&= \frac{4\pi}{3}\frac{q^2}{r}\gamma^4.
\end{aligned} \tag{94}$$

The $dI/d\omega$ angular-integrated spectrum was derived by Schwinger * to be:

$$\frac{dI}{d\omega} = \sqrt{3}\frac{q^2}{c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{+\infty} dx K_{5/3}(x). \tag{95}$$



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Case of periodic circular motion:
The results derived in the previous pages pertains to instantaneous circular motion, for which the spectrum is a continuum. If the motion is periodic, the associated spectrum is discrete. The tool for analyzing this type of motion are the Fourier series. First we note that the period measured by an observed in the far field (T) is the same as the period of the particle motion (T'). We now have to introduce the Fourier series decomposition:

$$\overrightarrow{A}(t) = \sqrt{\frac{c}{4\pi}}[E\overrightarrow{E}]_{ext} = \sum_{n=-\infty}^{n=+\infty} \overrightarrow{A}_n e^{-in\omega_0 t},$$

where,

$$\overrightarrow{A}_n = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \overrightarrow{A}(t) e^{in\omega_0 t}$$

Following what we did previously we can show:

$$\overrightarrow{A}_n = \sqrt{2\pi}\frac{q}{2\pi\sqrt{2c}}\frac{\omega_0}{2\pi}(-in\omega_0)\int_0^{2\pi/\omega_0} dt\hat{n}\times(\hat{n}\times\overrightarrow{\beta})e^{in\omega_0(t-\hat{n}.\overrightarrow{r})}$$

where the $\sqrt{2\pi}$ come from the difference in normalization factor between the Fourier integral transform and series.
The spectrum is now discrete $\omega = n\omega$ with $n \in \mathbb{N}$.

Thomson & Compton Scattering

"Scattering" of an e.m. wave by a charged particle (say e-). But e- has no surface $\rightarrow$ radiation is not really scattered. Radiation emitted by the e- as it oscillates in the incoming radiation field is the "scattered" radiation.
In term of photon: photon with wavelength λ strikes a stationary e- and bounce off with wavelength λ'.

$$\begin{aligned} P^\alpha_\gamma + P^\alpha_\gamma &= P^\alpha_{\gamma'} + P^\alpha_{e-'} \quad (96) \\ P^\alpha_{e-'} &= P^\alpha_{e-} + P^\alpha_\gamma - P^\alpha_{\gamma'} \quad (97) \end{aligned}$$

So the norm is:

$$m^2c^2 = (P^\alpha_{e-} + P^\alpha_\gamma - P^\alpha_{\gamma'})(P_{e-,\alpha} + P_{\gamma,\alpha} - P_{\gamma',\alpha}) \quad (98)$$

remembering that for a photon $P_\mu P^\mu = 0$ we finally end up with

$$P^\alpha_{\gamma'} P_{e-,\alpha} - P^\alpha_{\gamma'} P_{e-,\alpha} - P^\alpha_{\gamma'} P_{e-,\alpha} = 0 \quad (99)$$

we have:

$$P_{e-,\alpha} = mc(E_\gamma/2) - \overrightarrow{p}_{e-}.\overrightarrow{p}_\gamma = mE_\gamma \quad (100)$$

$$P^\alpha_{\gamma'}P_{\gamma,\alpha} = \frac{E_\gamma}{c}\frac{E_{\gamma'}}{c} - \overrightarrow{p}_\gamma.\overrightarrow{p}_{\gamma'} = -p_\gamma p_{\gamma'}\cos\theta + \frac{E_\gamma E_{\gamma'}}{c^2} \quad (101)$$

$$P_{e-,\alpha}P^\alpha_{\gamma'} = mE_\gamma. \quad (102)$$

Taking $E_\gamma \equiv \frac{hc}{\lambda}$ (and similarly for γ') we finally obtain:

$$\lambda - \lambda' = \frac{h}{mc}(1 - \cos\theta). \quad (103)$$

This is the usual Compton scattering. Thomson scattering is the non relativistic limit of Compton scattering (so take $c \to \infty$) so $\lambda = \lambda'$.

Cross section for Thomson scattering:

The cross-section is defined as:

$$\sigma \equiv \frac{\text{E radiated/time/solid angle}}{\text{incident flux/unit area/time}}. \quad (104)$$

e- is at rest $\overrightarrow{\beta} = 0$, and

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]\right|^2}{\kappa^5} \tag{105}$$

$$\xrightarrow{\beta\to 0} \frac{e^2}{4\pi c}\frac{\left|\hat{n}\times[\hat{n}\times\overrightarrow{\dot{\beta}}]\right|}{\kappa^5} = \frac{e^2}{4\pi c}\dot{\beta}^2\sin^2\Theta \tag{106}$$

where $\Theta\angle(\hat{n},\overrightarrow{\dot{\beta}})$. Introducing the acceleration $\overrightarrow{a}\equiv c\overrightarrow{\dot{\beta}}$ we have:

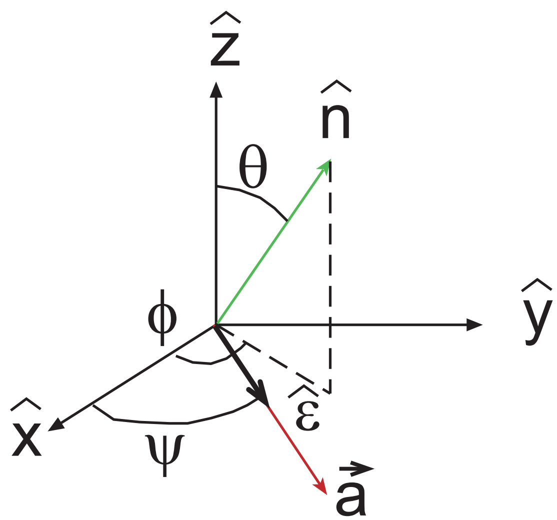
$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3}\sin^2\Theta. \tag{107}$$

Also note that in the NR limit $t\to t'$ so $\frac{dP(t)}{d\Omega} = \frac{dP(t')}{d\Omega}$. We now need to find $\overrightarrow{a}$.

Let's consider an incoming plane e.m. wave of the form $\overrightarrow{E}(\overrightarrow{x}, t) = \widehat{\epsilon} E_0 e^{i(\overrightarrow{k} . \overrightarrow{x} - \omega t)}$ then we have

$$\overrightarrow{a}(t) = \frac{e}{m} \widehat{\epsilon} E_0 e^{i(\overrightarrow{k} . \overrightarrow{x} - \omega t)} \tag{108}$$

we only consider the E-field contribution since $\beta = 0$. Let $\overrightarrow{k} = k\widehat{z}$.



From the figure we have:

$$\begin{aligned}
\hat{\epsilon} &= \cos\psi\hat{x} + \sin\psi\hat{y} && (109)\\
\hat{n} &= \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} && (110)\\
\hat{n}.\overrightarrow{a} &= as_\theta(c_\psi c_\phi + s_\psi s_\phi) = as_\theta c_{\psi-\phi} && (111)\\
&= a\sin\theta\cos(\psi-\phi) = a\cos\Theta. && (112)
\end{aligned}$$

Thus

$$\sin^2\Theta = 1 - \sin^2\theta\cos^2(\psi-\phi). \tag{113}$$

t-averaged emitted power scales as $\langle a^2(t)\rangle_t$.

$$\langle a^2\sin^2\Theta\rangle_t = \frac{1}{2}\left(\frac{eE_0}{m}\right)^2[1-\sin^2\theta\cos^2(\psi-\phi)]. \tag{114}$$

If incident radiation is not polarized:

$$\langle\cos^2(\psi-\phi)\sin^2\theta\rangle_\psi = \frac{1}{2}\sin^2\theta. \tag{115}$$

So

$$\langle a^2 \sin^2 \Theta \rangle_{t,\psi} = \frac{1}{2}\left(\frac{eE_0}{m}\right)^2 [1 - \frac{1}{2}\sin^2\theta] \tag{116}$$

So finally the radiated power per unit of solid angle takes the form:

$$\langle \frac{dP}{d\Omega} \rangle_{t,\psi} = \frac{cE_0^2}{16\pi}\left(\frac{e^2}{mc^2}\right)^2 [1 + \cos^2\theta] = \frac{c}{16\pi} r_e (1 + \cos^2\theta). \tag{117}$$

The incoming Poyinting flux is:

$$S \;=\; \frac{c}{8\pi}\overrightarrow{E} \times \overrightarrow{H}^* \tag{118}$$

the time average power per unit area is:

$$\frac{dP}{d\sigma} = S = \frac{c}{8\pi}E_0^2. \tag{119}$$

So the cross-section is:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dP}\frac{dP}{d\Omega} = \frac{\frac{cr_e^2}{16\pi}E_0^2}{\frac{c}{8\pi}E_0^2}[1+\cos^2\theta] \tag{120}$$

$$= \frac{1}{2}r_e^2(1+\cos^2\theta) \tag{121}$$

This is Thomson scattering formula. The integrated cross-section is:

$$\sigma = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{d\sigma}{d\Omega} = \frac{16\pi}{3}\frac{1}{2}r_e^2 \tag{122}$$

$$= \frac{8\pi}{3}r_e^2. \tag{123}$$

Thomson and Compton scattering apply to a free-electron. Let's now consider a bounded electron whose dynamics is described as a damped oscillator model:

$$\vec{a} + \Gamma \vec{v} + \omega_0^2 \vec{x} = \frac{q}{m} \vec{E}. \tag{124}$$

As before consider $\vec{E} = \hat{\epsilon} E_0 e^{i(\vec{k}.\vec{x} - \omega t)}$, Take $\vec{x} = \vec{x}_0 e^{-i\omega t}$. We then have:

$$(-\omega^2 - i\omega\Gamma + \omega_0^2)\vec{x}_0 = \hat{\epsilon}\frac{q}{m} E_0 e^{i\vec{k}.\vec{x}} \tag{125}$$

assume $\vec{k}.\vec{x} = 0$, that is $|x| \ll \lambda$ (e- orbit is small compared to radiation wavelength). Then

$$\vec{x}_0 \simeq \frac{\frac{e}{m} E_0}{\omega_0^2 - \omega^2 - i\omega\Gamma} \hat{\epsilon}, \tag{126}$$

and

$$\vec{a} = -\omega^2 \vec{x} \Rightarrow |a^2| = \left(\frac{e}{m} E_0\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}. \tag{127}$$

$$|a^2| = \frac{\left(\frac{e}{m}E_0\right)^2}{\left(\frac{\omega_0}{\omega}\right)^2 + \left[\left(\frac{\Gamma}{\omega}\right)^2 - 1\right]^2}. \tag{128}$$

Same as before but modified to include $\overrightarrow{a}$'s denominator. So finally for a bounded e-, we get:

$$\langle\frac{dP}{d\Omega}\rangle = \frac{c}{16\pi} r_e^2 E_0^2 \frac{1+\cos^2\theta}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \left(\frac{\Gamma}{\omega}\right)^2}, \tag{129}$$

and the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \frac{1+\cos^2\theta}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \left(\frac{\Gamma}{\omega}\right)^2}, \tag{130}$$

The limit $\omega \gg \omega_0$, $\omega \gg \Gamma$ corresponds to Thomson scattering, while the limit $\omega \ll \omega_0$, $\omega \gg \Gamma$ gives the Rayleigh formula:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \left(\frac{\omega}{\omega_0}\right)^4 [1 + \cos^2\theta] \propto \omega^4 \tag{131}$$

So high frequencies are scattered more preferably than low frequencies. This explains why the sky is blue...