– Complex Numbers: What you need to know for PHYS 375 –

1 Introduction

Although other notations can be used, complex numbers are very often written in the form :

$$a + bi$$

where a and b are real numbers, and i is the imaginary unit, which has the property $i^2 = -1$. The real number a is called the 'real part of the complex number, and the real number b is the imaginary part.

Real numbers may be expressed as complex numbers with the imaginary part of zero; that is, the real number a is equivalent to the complex number $a + 0i$. Complex numbers with a real part which is zero are called imaginary numbers.

For example, $3+2i$ is a complex number, with real part 3 and imaginary part 2. If $z = a+ib$, the real part a is formally denoted $Re(z)$ or $\Re(z)$, and the imaginary part b is denoted $Im(z)$ or $\Im(z)$.

In some disciplines (in particular, electrical engineering, where i is a symbol for Electric current), the imaginary unit i is instead written as j, so complex numbers are sometimes written as $a + jb$.

The set of all complex numbers is usually denoted $\mathbb{C}$. The real numbers ensemble, $\mathbb{R}$, may be regarded as a subset of $\mathbb{C}$ by considering every real number as a complex i.e. $a = a + 0i$.

2 Properties

Two complex numbers are equal iff if their real parts are equal and their imaginary parts are equal. That is,

$$a + ib = c + id \leftrightarrow a = b \text{ and } c = d.$$

Complex numbers are added, subtracted, multiplied, and divided by formally applying the associative, commutative and distributive laws of algebra, together with the equation $i^2 = -1$

Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Subtraction:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Multiplication:

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

Division:

$$\frac{(a+bi)}{(c+di)} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right) i$$

To a complex number $z = a + ib$ we can associate its complex conjugate $\bar{z} = a - ib$. The complex conjugate is also sometime noted z^* or c.c.(z).

3 Representation

A complex number z can be viewed as a point or a vector in a two-dimensional Cartesian coordinate system called the "complex plane" or "Argand diagram"; see Figure 1. The point and hence the complex number z can be specified by Cartesian (rectangular) coordinates. The Cartesian coordinates of the complex number are the real part $x = \mathrm{Re}(z)$ and the imaginary part $y = \mathrm{Im}(z)$. The representation of a complex number by its Cartesian coordinates is called the "Cartesian form" or "rectangular form" or "algebraic form" of that complex number.

The "'absolute value"' (or "modulus" or "magnitude") of a complex number $z = re^{i\phi}$ is defined as $|z| = r$. Algebraically, if $z = x + yi$, then $|z| = \sqrt{x^2 + y^2}$. Note that the square of the modulus is simply obtained by multiplying the complex number with its complex conjugate number

One can check readily that the absolute value has three important properties:

$$|z| = 0$$

if and only if

$$z = 0$$

$$|z + w| \leq |z| + |w|$$

(triangle inequality)

$$|z \cdot w| = |z| \cdot |w|$$

for all complex numbers z and w. It then follows, for example, that

$$|1| = 1$$

and

$$|z/w| = |z|/|w|$$

.

Alternatively to the cartesian representation $z = x + iy$, the complex number z can be specified by Polar coordinate system. The polar coordinates are r, called the Absolute value or modulus, and ϕ (also noted $\arg(z)$), called the argument or the angle of z. The conversion from the polar form to the Cartesian form follows as

$$x = r\cos\varphi$$

$$y = r \sin \varphi$$

and the reciprocal conversion is

$$r = \sqrt{x^2 + y^2}$$

$$\varphi = \arg(z) = \arctan(y, x)$$

The notation of the polar form as $z = r\left(\cos \varphi + i \sin \varphi\right)$ is called trigonometric form. Using Euler's formula it can also be written as

$$z = r\, \mathrm{e}^{i\varphi}$$

which is called "exponential form".

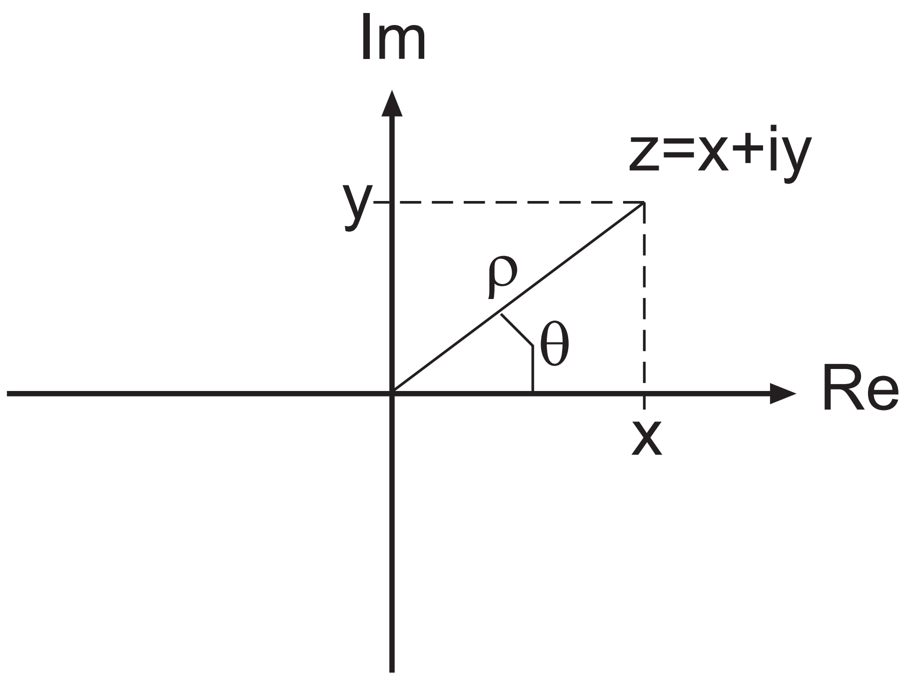

Figure 1: Cartesian and polar representation of a complex number $z = x + iy$.

4 Credit

This document was compiled by borrowing text/equations from Eric Weisstein's MathWorld and Wikipedia.