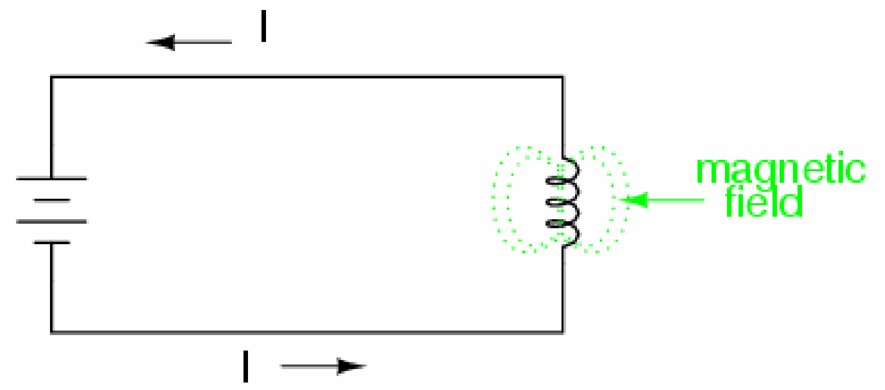
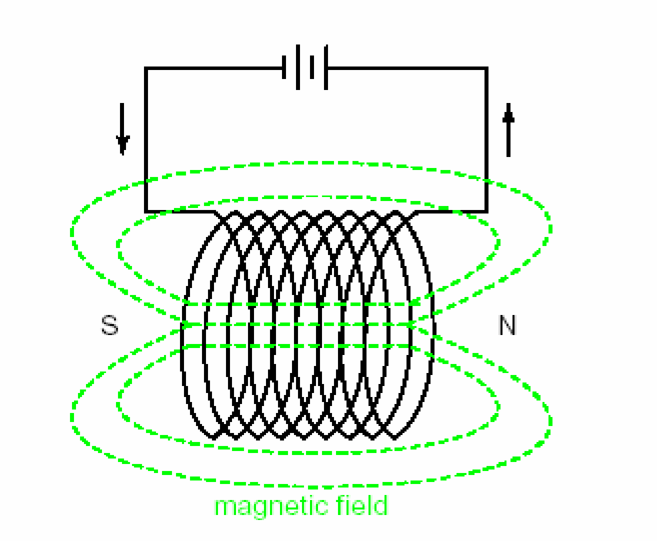
Lesson 3: RLC circuits & resonance

- Inductor, Inductance
- Comparison of Inductance and Capacitance
- Inductance in an AC signals
- RL circuits
- LC circuits: the electric “pendulum”
- RLC series & parallel circuits
- Resonance

Inductor





- Start with Maxwell's equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- Integrate over a surface S (bounded by contour C) and use Stoke's theorem:

Magnetic flux in Weber

$$\iint_S \vec{\nabla} \times \vec{E}.d\vec{A} = \oint_{S \in C} \vec{E}.d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t}.d\vec{A} = -\frac{\partial \Phi}{\partial t}$$

- The voltage is thus

$$V_L = -emf = \frac{\partial \Phi}{\partial t}$$

Wihelm Weber (1804-1891)

Inductor

- Now need to find a relation between magnetic field generated by a loop and current flowing through the loop's wire. Used Biot and Savart's law:

$$d\vec{B} = \frac{\mu_0}{4\pi} I d\vec{l} \times \frac{\hat{r}}{r^2} \Rightarrow B \propto I$$

- Integrate over a surface S the magnetic flux is going to be of the form

$$\Phi \equiv LI$$

Inductance measured in Henri (symbol H)

- The voltage is thus

$$V_L = \frac{\partial \Phi}{\partial t} = L\frac{dI}{dt}$$

Joseph Henri (1797-1878)

Inductor

- Case of loop made with an infinitely thin wire

$$B = \frac{\mu}{4\pi}\delta l.I$$

- If the inductor is composed of n loop per meter then total B-field is

$$B = \frac{\mu}{4\pi} nI$$

- So inductance is

Increase magnetic permeability (e.g. use metallic core instead of air)

$$\Phi \equiv BA = \frac{\mu}{4\pi} AnI \Rightarrow L = \frac{\mu}{4\pi} An$$

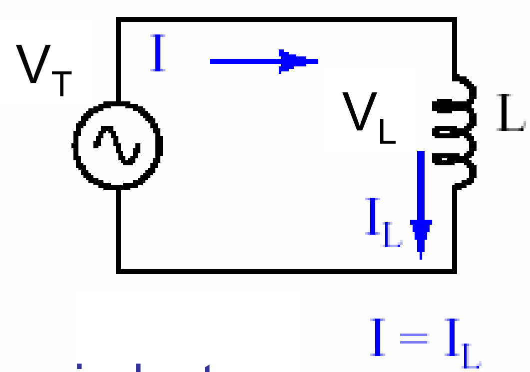
Increase number of wire per unit length increase L

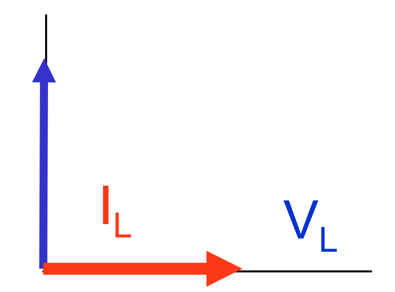
Area of the loop

Inductor in an AC Circuit

$$V_L = L\frac{dI}{dT}$$

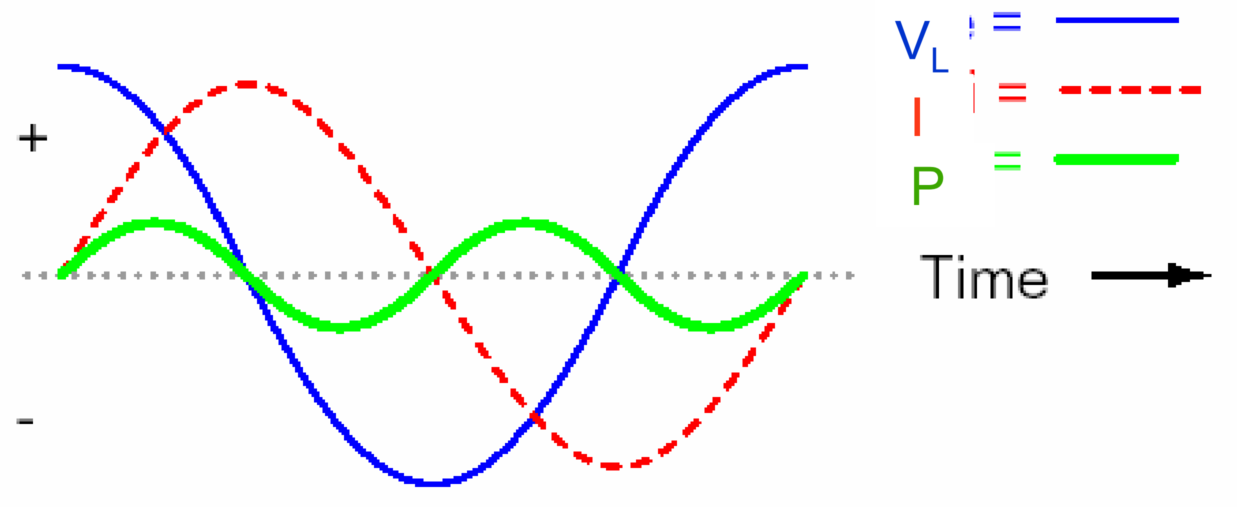
$$\Rightarrow Z = \frac{V_L}{I} = i\omega L$$





- Introduce reactance for an inductor:

$$X_L = \omega L$$



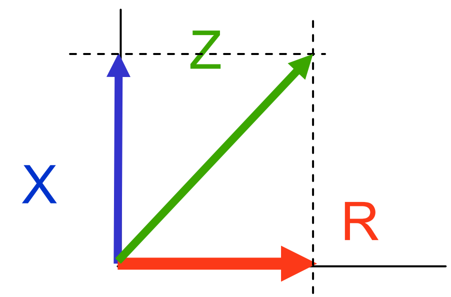
Inductor , Capacitor, Resistor

- Resistance = friction against motion of electrons

- Reactance = inertia that opposes motion of electrons

$$X_L = \omega L$$

$$X_C = -\frac{1}{\omega C}$$



- Impedance is a generally complex number:

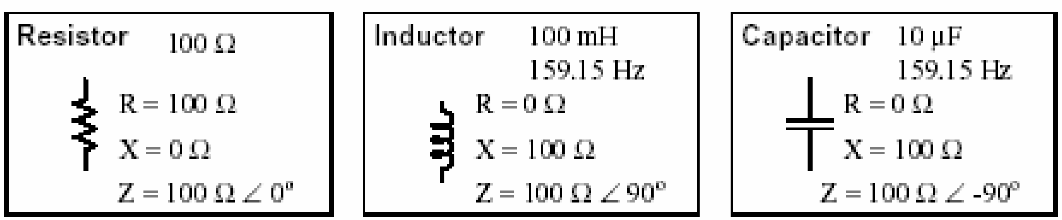
$$Z = R + iX$$

- Note also one introduces the Admittance:

$$Y = \frac{1}{Z} = G + iB$$

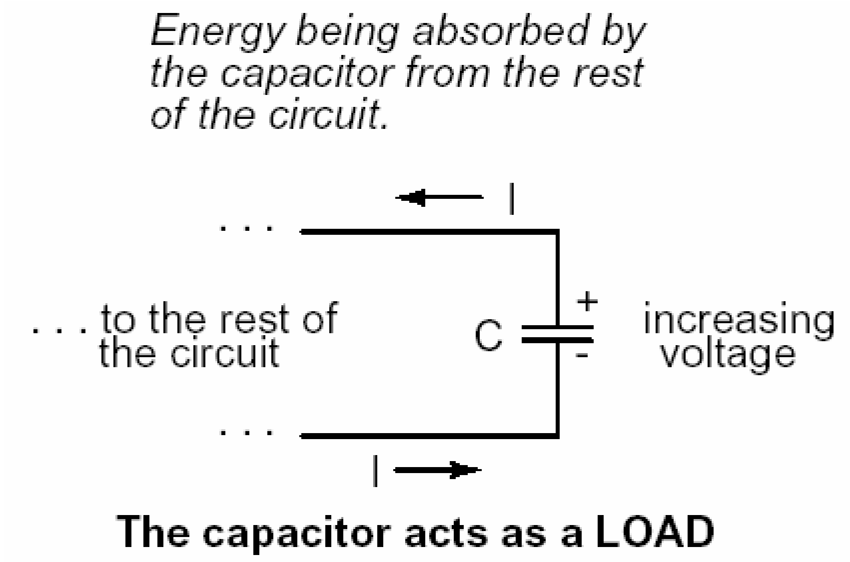
susceptance

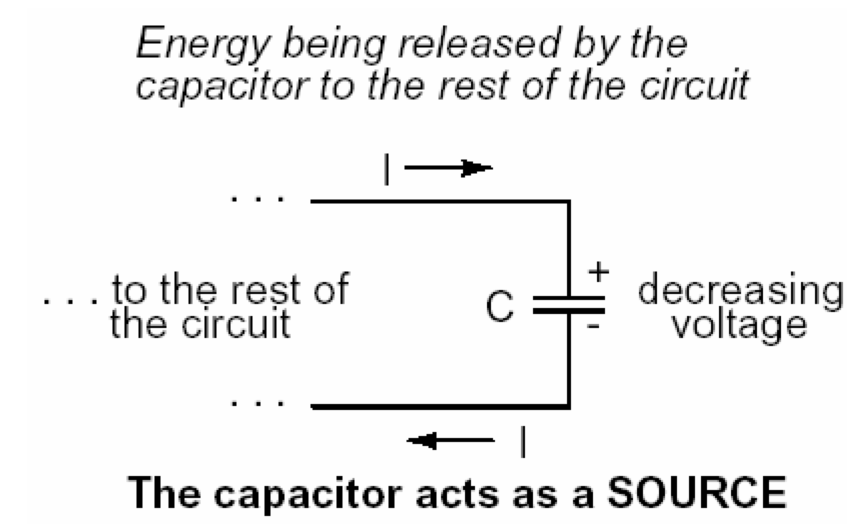
conductance



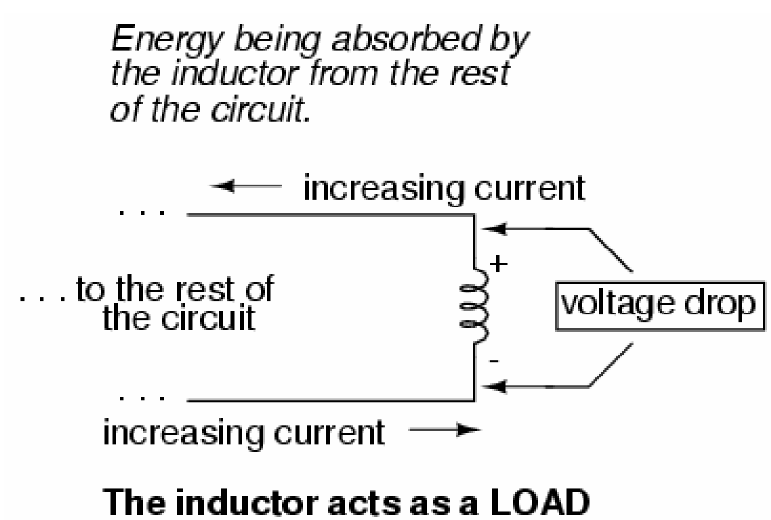
Inductor versus Capacitor

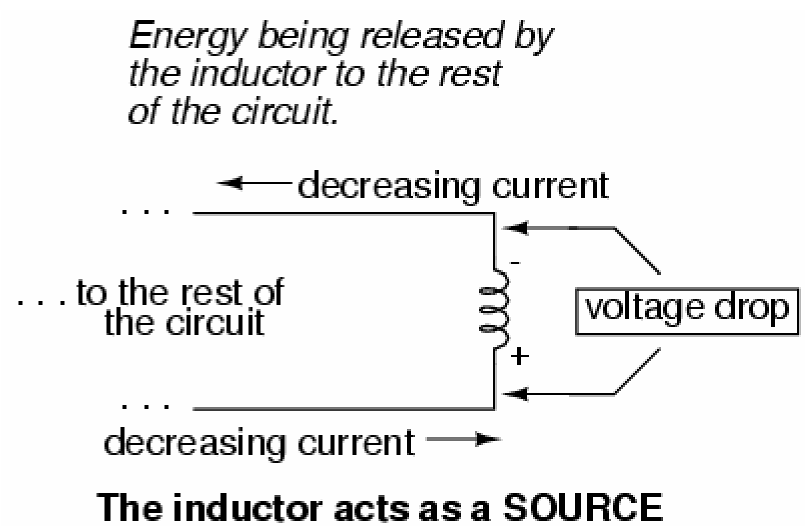
CAPACITOR





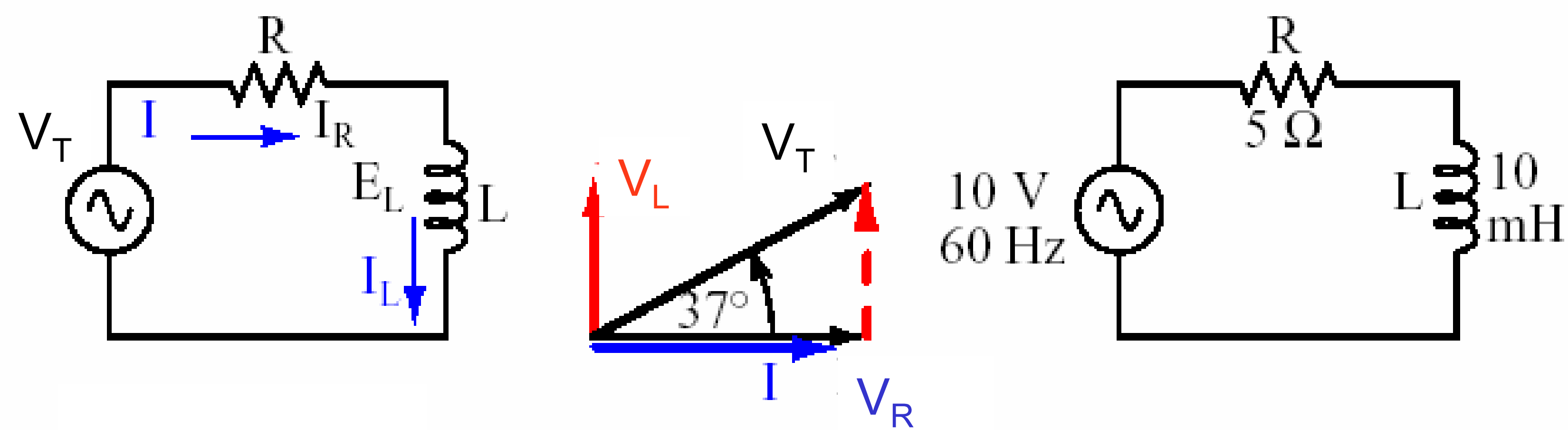
INDUCTOR





RL series Circuits

$$V_T = V_R + V_L = RI + L\frac{dI}{dt} = (R + i\omega L)I \Rightarrow Z = R + i\omega L = R + iX_L$$



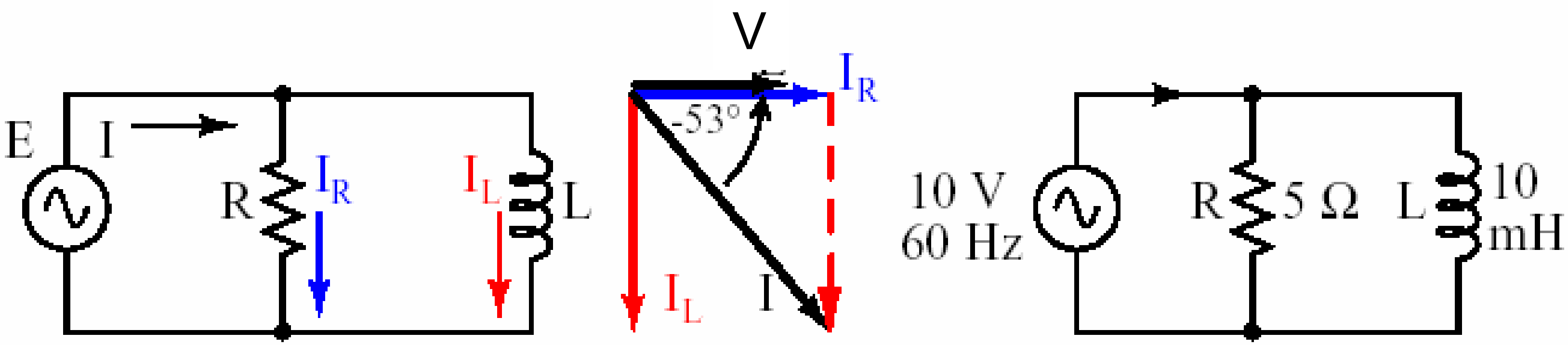
- For the above circuit we can compute a numerical value for the impedance:

$$Z = (5 + 3.7699i) \quad \Omega$$

$$|Z| = \sqrt{5^2 + 3.7699^2} \approx 6.262, \quad \Theta = 37.02°$$

RL parallel Circuits

$$I = I_R + I_L = \frac{V}{R} + \frac{1}{L}\int V dt = (\frac{1}{R} + \frac{1}{i\omega L})V \Rightarrow Z^{-1} = R^{-1} + (i\omega L)^{-1}$$



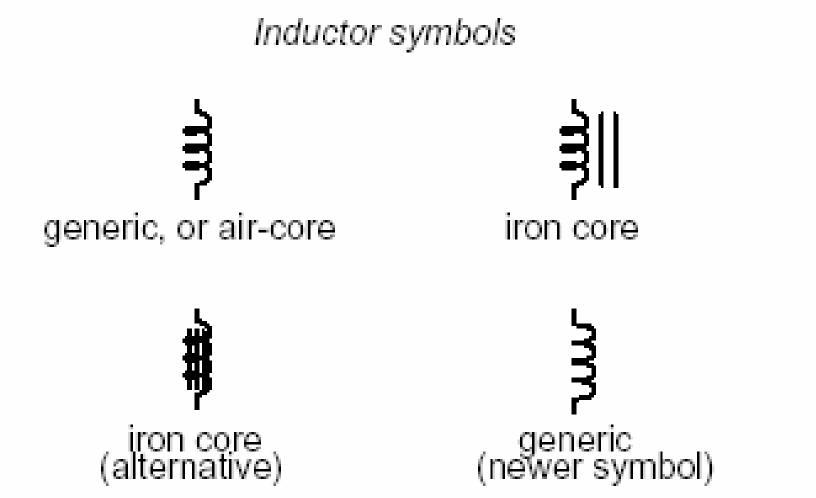
- For the above circuit we can compute a numerical value for the impedance:

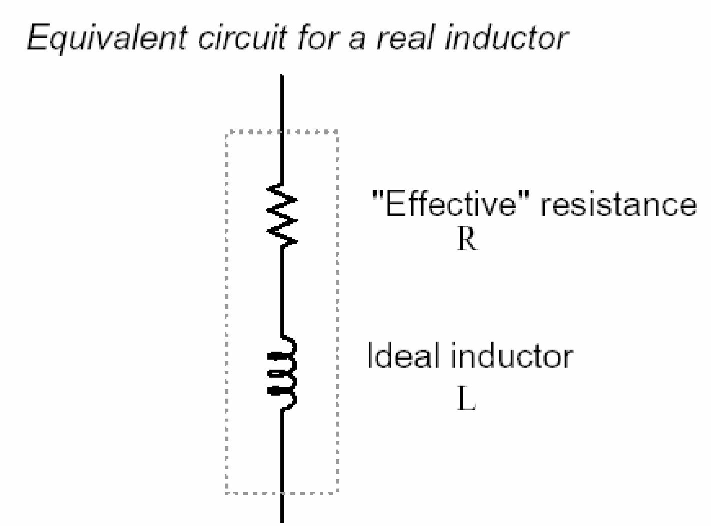
$$Z = (1.81 + 2.40i) \quad \Omega$$

$$|Z| \approx 3.01, \quad \Theta = 52.98°$$

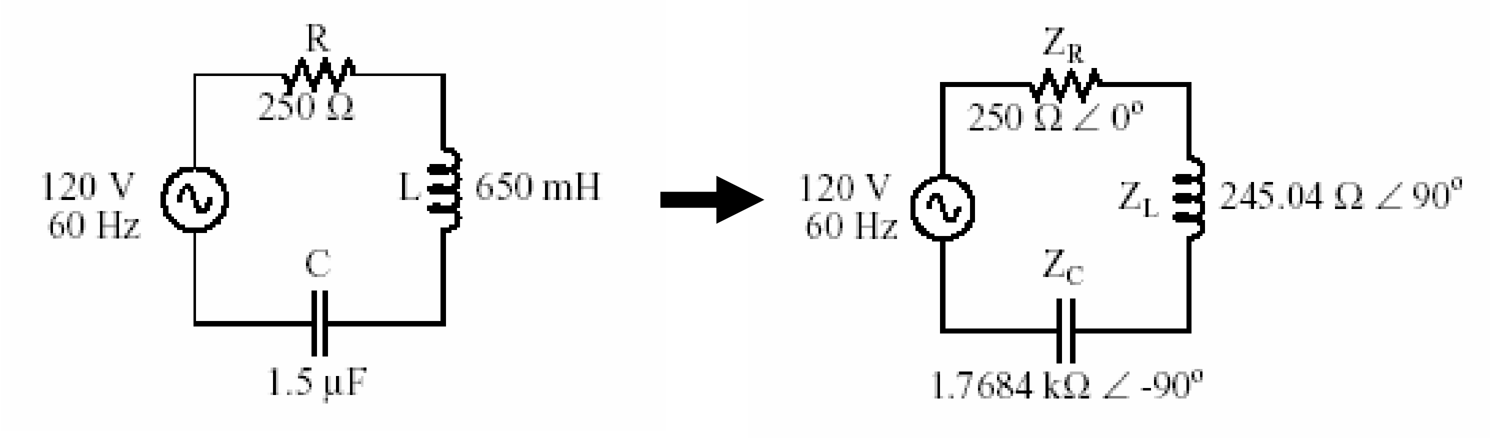
Inductor: Technical aspects

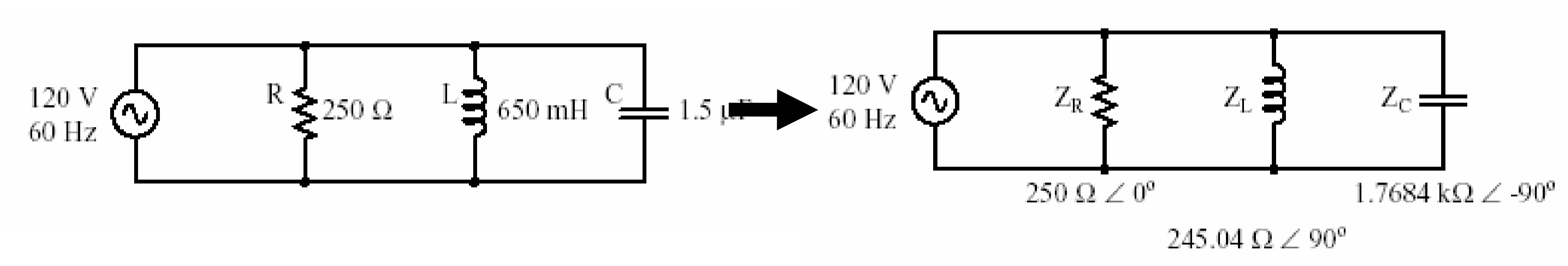
- Inductors are made a conductor wired around air or a ferromagnetic core
- Unit of inductance is Henri, symbol is H
- Real inductors also have a resistance (in series with inductance)





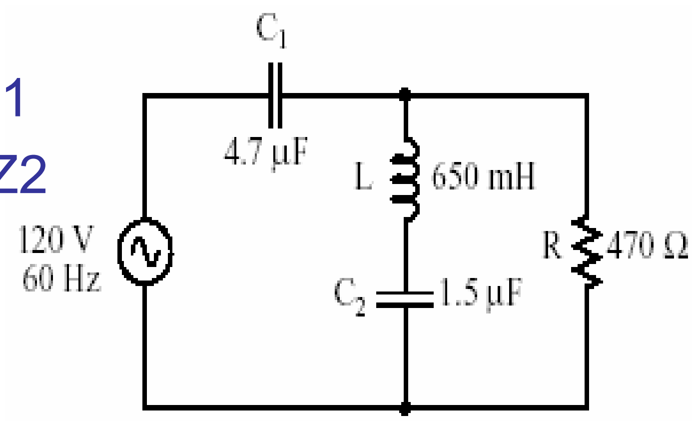
RLC series/parallel Circuits





RLC series/parallel Circuits: an example

- Compute impedance of the circuit below
  - Step 1: consider C2 in series with L ⇨ Z1
  - Step 2: consider Z1 in parallel with R ⇨ Z2
  - Step 3: consider Z2 in series with C



- Let's do this:

$$Z_1 = i\left(L\omega - \frac{1}{C\omega}\right) = 1523.34i \qquad Z_2 = \frac{1}{\dfrac{1}{Z_1} + \dfrac{1}{R}} = 429.15 - 132.41i$$

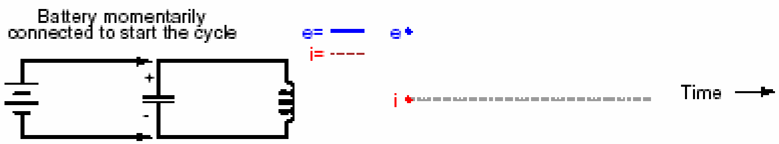
$$Z_3 = Z_2 - \frac{i}{C_1\omega} = 429.15 - 629.79i$$

- Current in the circuit is

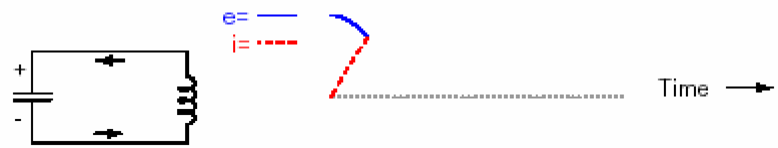
$$I = \frac{V}{Z_3} = 76.89 + 124.86i \Rightarrow |I| = 146.64\ mA,\ \angle I = 58.371°$$

- And then one can get the voltage across any components

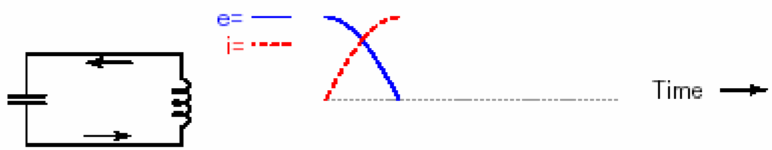
LC circuit: An electrical pendulum



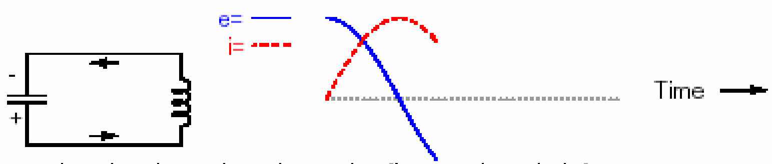
capacitor charged: voltage at (+) peak
inductor discharged: zero current



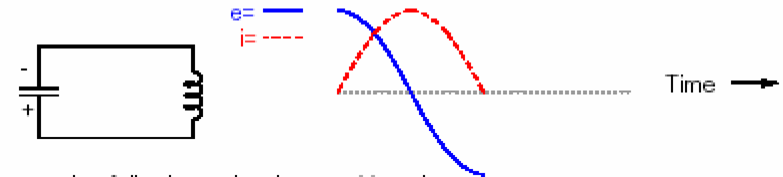
capacitor discharging: voltage decreasing
inductor charging: current increasing



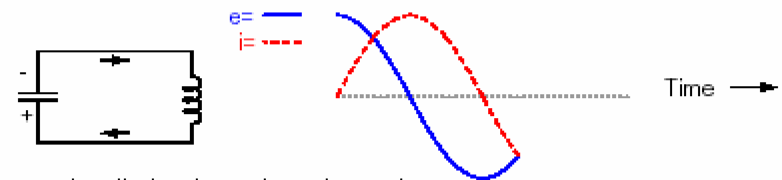
capacitor fully discharged: zero voltage
inductor fully charged: maximum current



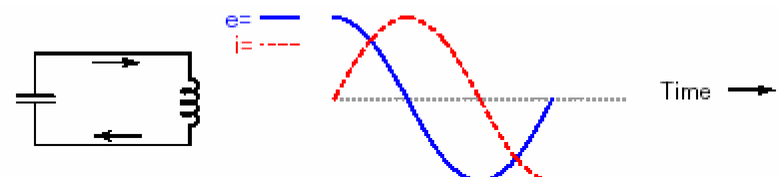
capacitor charging: voltage increasing (in opposite polarity)
inductor discharging: current decreasing



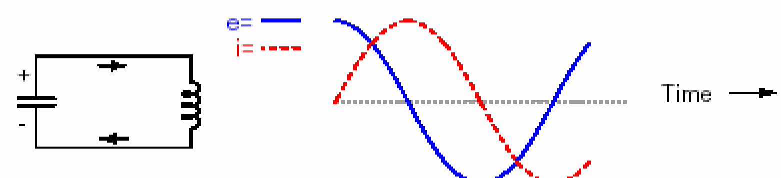
capacitor fully charged: voltage at (-) peak
inductor fully discharged: zero current



capacitor discharging: voltage decreasing
inductor charging: current increasing



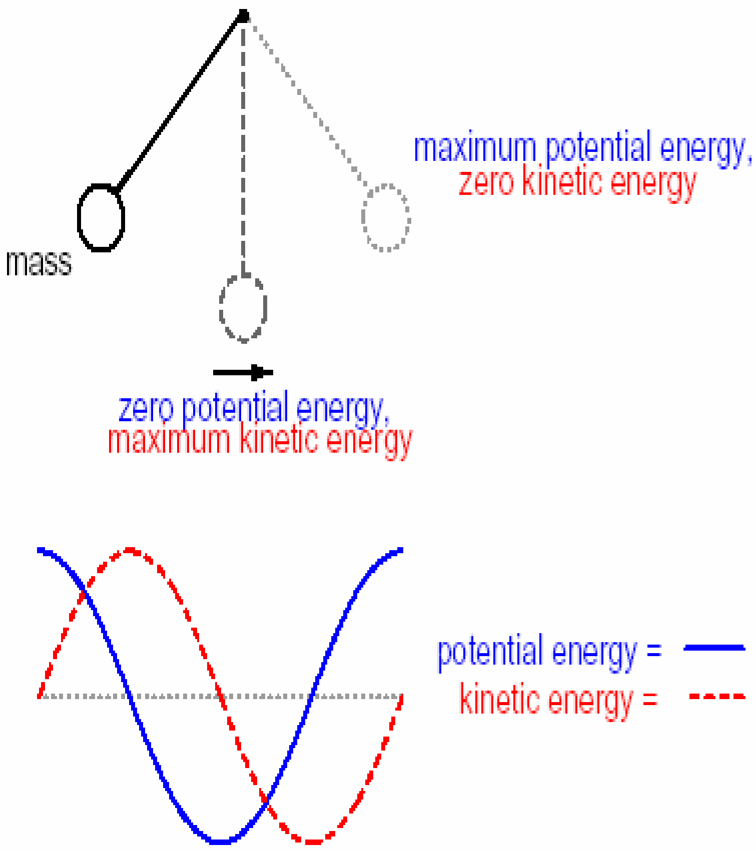
capacitor fully discharged: zero voltage
inductor fully charged: current at (-) peak



capacitor charging: voltage increasing
inductor discharging: current decreasing

LC circuit: An electrical pendulum

- Mechanical pendulum: oscillation between potential and kinetic energy
- Electrical pendulum: oscillation between magnetic ($1/2LI^2$) and electrostatic ($1/2CV^2$) energy
- In practice, the LC circuit showed has some resistance, i.e. some energy is dissipated and therefore the oscillation amplitude is damped. The oscillation frequency keeps unchanged.
- LC circuit are sometime called tank circuit and oscillate (=resonate)

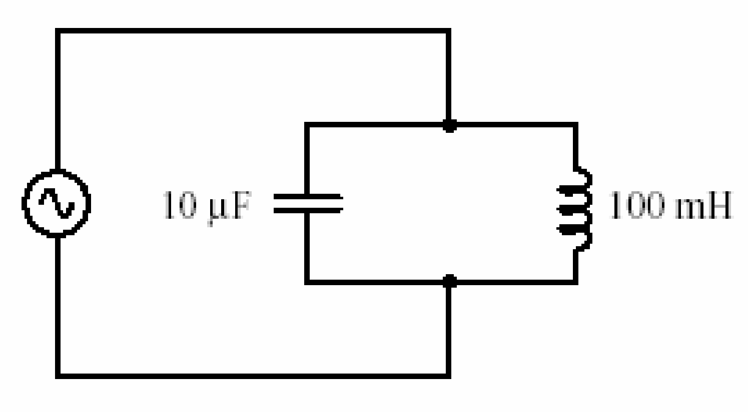


Example of a simple "tank" (LC) circuit

- ODE governing this circuit?

$$I = I_C + I_L = C\frac{dV}{dt} + \frac{1}{L}\int V dt$$

$$\frac{dI}{dt} = C\frac{d^2V}{dt^2} + \frac{1}{L}V \Leftrightarrow \boxed{\frac{d^2V}{dt^2} + \frac{1}{LC}V = \frac{dI}{dt}}$$



- Equation of a simple harmonic oscillator with pulsation:

$$\omega = \frac{1}{\sqrt{LC}}$$

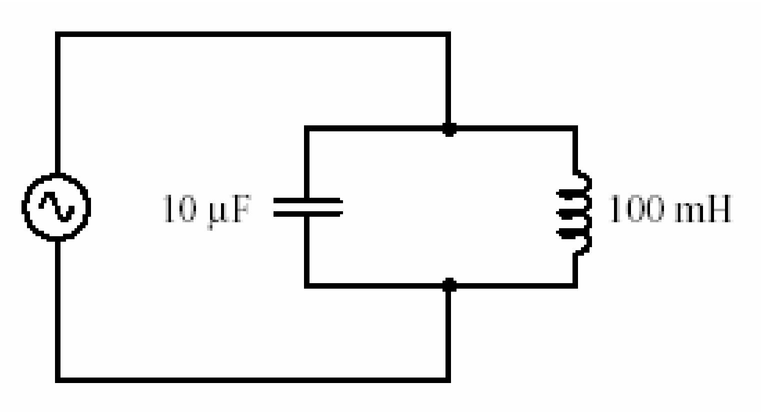
- Or one can state that system oscillate if impedance associated to C and L are equal, i.e.:

$$L\omega = \frac{1}{C\omega}$$

Example of a simple "tank" (LC) circuit

- What is the total impedance of the circuits?

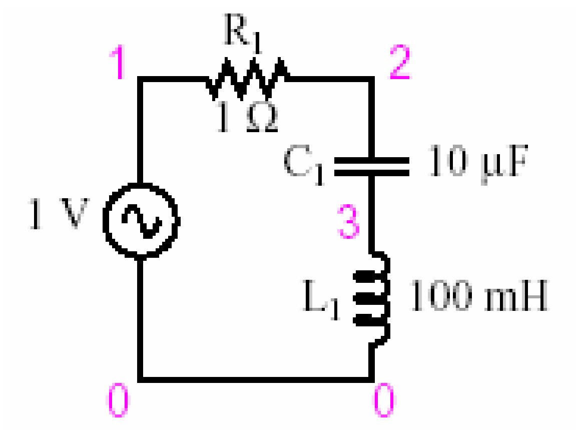
$$\boldsymbol{Z}^{-1} = \boldsymbol{i}\boldsymbol{L}\omega - \frac{\boldsymbol{i}}{\boldsymbol{C}\omega} = \boldsymbol{i}(\boldsymbol{L}\omega - \boldsymbol{L}\omega) = 0 \Rightarrow \boldsymbol{Z} = \infty$$



- So the tank circuit behaves as an open circuit at resonance!

- In a very similar way one can show that a series LC circuit behaves as a short circuit when driven on resonance i.e.,

$$Z = 0 \longrightarrow$$



RLC series circuit

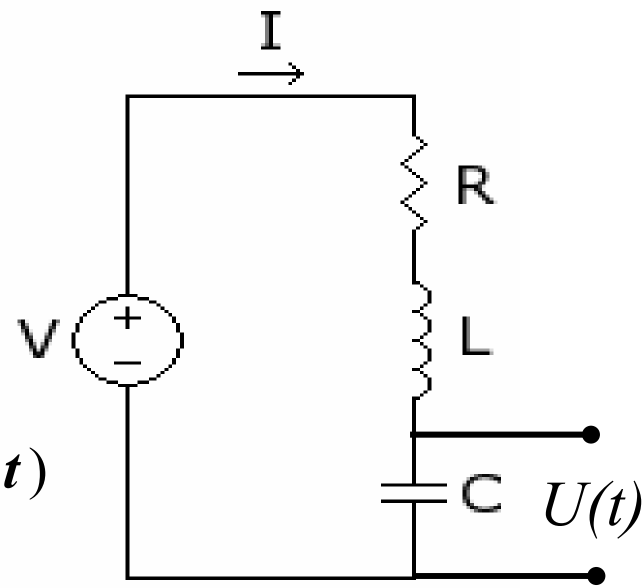
- 2^{nd} order ODE:

$$RI + L\frac{dI}{dt} + \frac{Q}{C} = V$$

$$with \quad I = \frac{dQ}{dt};\ Q = CU$$

Voltage across capacitor

$$\Rightarrow LC\frac{d^2U(t)}{dt^2} + RC\frac{dU(t)}{dt} + U(t) = V(t)$$



- Resonant frequency still $\omega_0 = \frac{1}{\sqrt{LC}}$

- Let's define the parameter $\zeta = \frac{R}{2L}$

- Then the ODE rewrites $$\frac{d^2U}{dt^2} + 2\zeta\frac{dU}{dt} + \omega_0^2 U = V$$

RLC series circuit: regimes of operation (1)

- Let's consider V(t) to be a dirac-like impulsion (not physical…) at t=0. Then for t>0, V(t)=0 and the previous equation simplifies to

$$\frac{d^2U}{dt^2} + 2\zeta\frac{dU}{dt} + \omega_0^2 U = 0$$

- With solutions

$$U(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

- Where the λ are solutions of the characteristics polynomial is

$$\lambda^2 + 2\zeta\lambda + \omega_0^2 = 0$$

- The discriminant is

$$\Delta = R^2C^2 - 4LC$$

- And the solutions are

$$\lambda_\pm = \frac{1}{2}\left(-2\zeta \pm \sqrt{\Delta}\right)$$

RLC series circuit: regimes of operation (2)

- If Δ<0 that is if $R < 2\sqrt{\dfrac{L}{C}}$

  Under damped

$$\lambda_{\pm} = -\frac{R}{2L} \pm \left[\left(\frac{R}{2L} \right)^2 - \frac{1}{LC} \right]^{1/2} \equiv -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

U(t) is of the form

$$\boxed{U(t) = e^{-\delta t}\left[Ae^{\sqrt{\delta^2-\omega_0^2}\,t} + Be^{-\sqrt{\delta^2-\omega_0^2}\,t} \right]}$$

A and *B* are found from initial conditions.

- If Δ=0 critical damping

$$\boxed{U(t) = Ae^{-\delta t}}$$

RLC series circuit: regimes of operation (3)

- If Δ>0 that is if $R > 2\sqrt{\frac{L}{C}}$

  Strong damping

$$\lambda_{\pm} = -\frac{R}{2L} \pm i\left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} \equiv -\delta \pm i\sqrt{\omega_0^2 - \delta^2}$$

U(t) is of the form

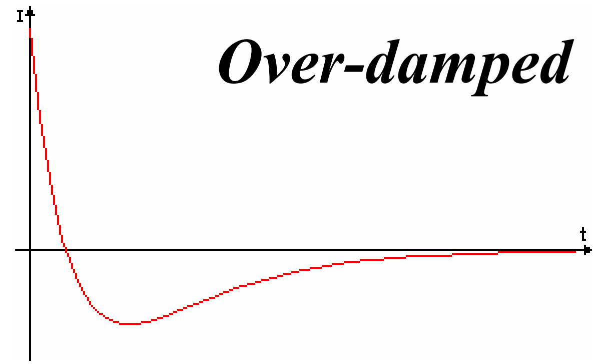
$$\boxed{U(t) = e^{-\delta t}\left[Ae^{i\sqrt{\omega_0^2 - \delta^2}\,t} + Be^{-i\sqrt{\omega_0^2 - \delta^2}\,t}\right]}$$

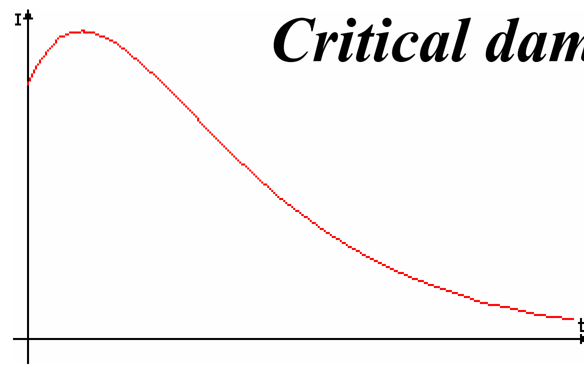
A and *B* are found from initial conditions.

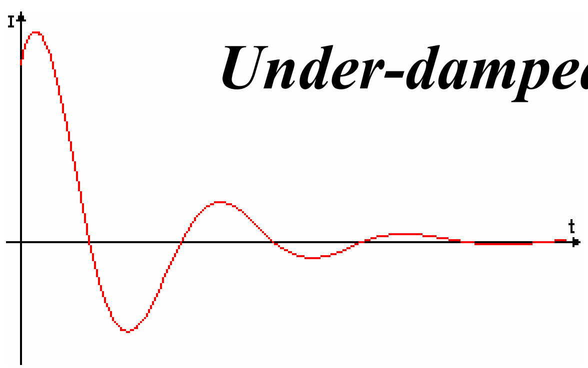
Which can be rewritten

$$\boxed{U(t) = De^{-\delta t}\sin\left(\sqrt{\omega_0^2 - \delta^2}\,t + \phi\right)}$$

RLC series circuit: regimes of operation (4)







- For **under-damped regime**, the solutions are exponentially decaying sinusoidal signals. The time requires for these oscillation to die out is 1/Q where the quality factor is defined as:

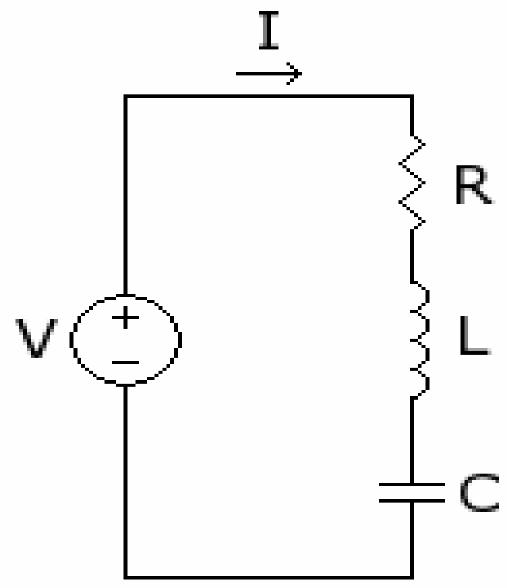
$$Q \equiv \frac{1}{R}\sqrt{\frac{L}{C}}$$

RLC series circuit: Impedance (1)

- 2nd order ODE:

$$V = V_R + V_L + V_C = RI + L\frac{dI}{dt} + \frac{1}{C}\int I.dt$$

$$\Rightarrow \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dV}{dt}$$



- Resonant frequency still $\omega_0 = \frac{1}{\sqrt{LC}}$
- Let's define the parameter $\zeta = \frac{R}{2L}$
- Then the ODE rewrites

$$\frac{d^2I}{dt^2} + 2\zeta\frac{dI}{dt} + \omega_0^2 I = \frac{1}{L}\frac{dV}{dt}$$

RLC series circuit: Impedance (2)

- Take back (but could also jut compute the impedance of the system)

$$\frac{d^2 I}{dt^2} + 2\zeta \frac{dI}{dt} + \omega_0^2 I = \frac{1}{L}\frac{dV}{dt}$$

- Explicit *I* in its complex form and deduce the current:

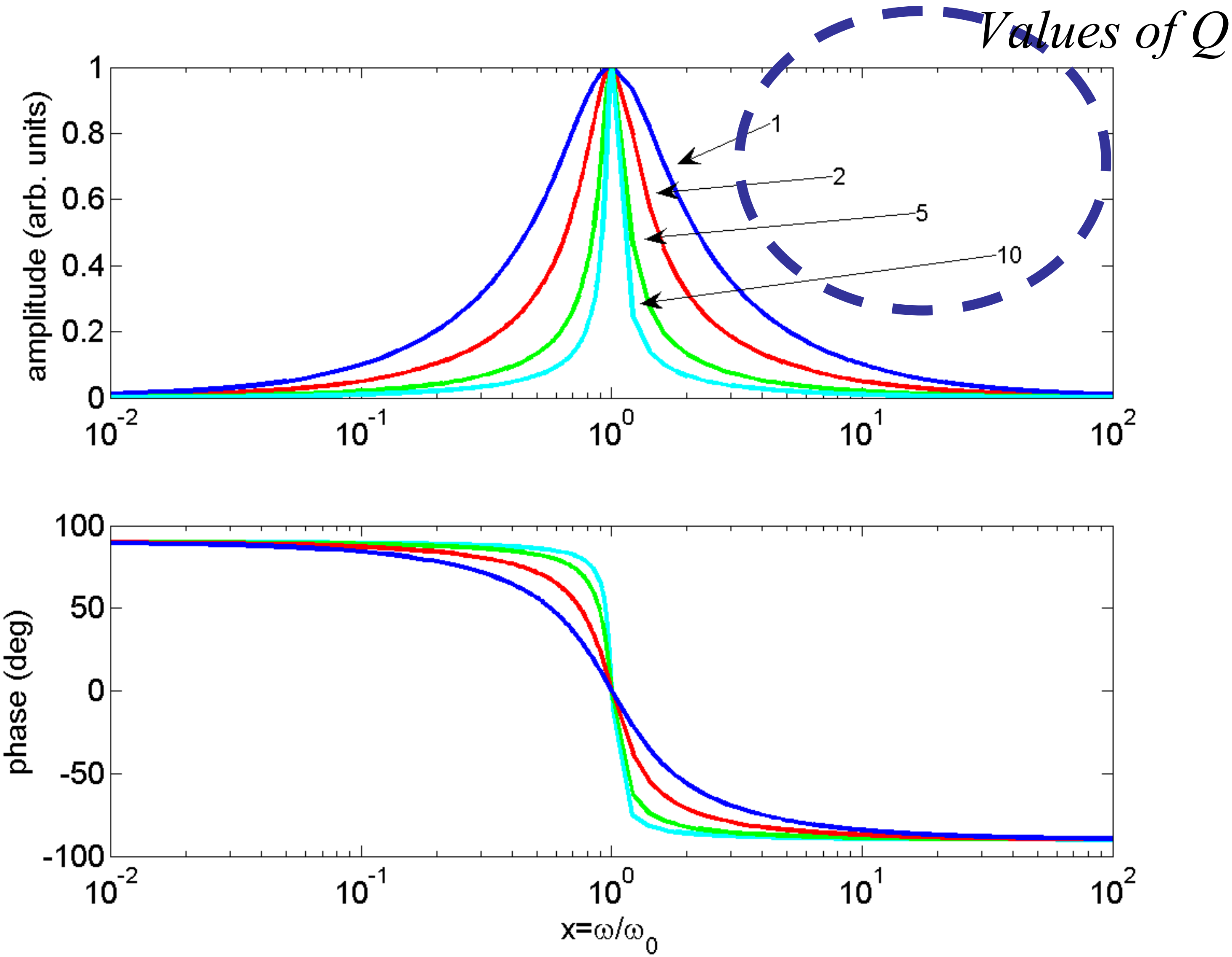
$$-\omega^2 I + 2i\zeta\omega I + \omega_0^2 I = i\frac{1}{L}\omega V$$

$$\Rightarrow Y \equiv \frac{I}{V} = \frac{i\omega}{\omega_0^2 - \omega^2 + 2i\zeta\omega} \Rightarrow |Y| = \frac{1}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

- Introducing $x=\omega/\omega_0$ we have

$$|Y| = \frac{1}{R\sqrt{1 + Q^2\left(x - \frac{1}{x}\right)^2}}$$

RLC series circuit: resonance



RLC parallel circuit : resonance

- The same formalism as before can be applied to parallel RLC circuits.
- The difference with serial circuit is: at resonance the impedance has a maximum (and not the admittance as in a serial circuit)