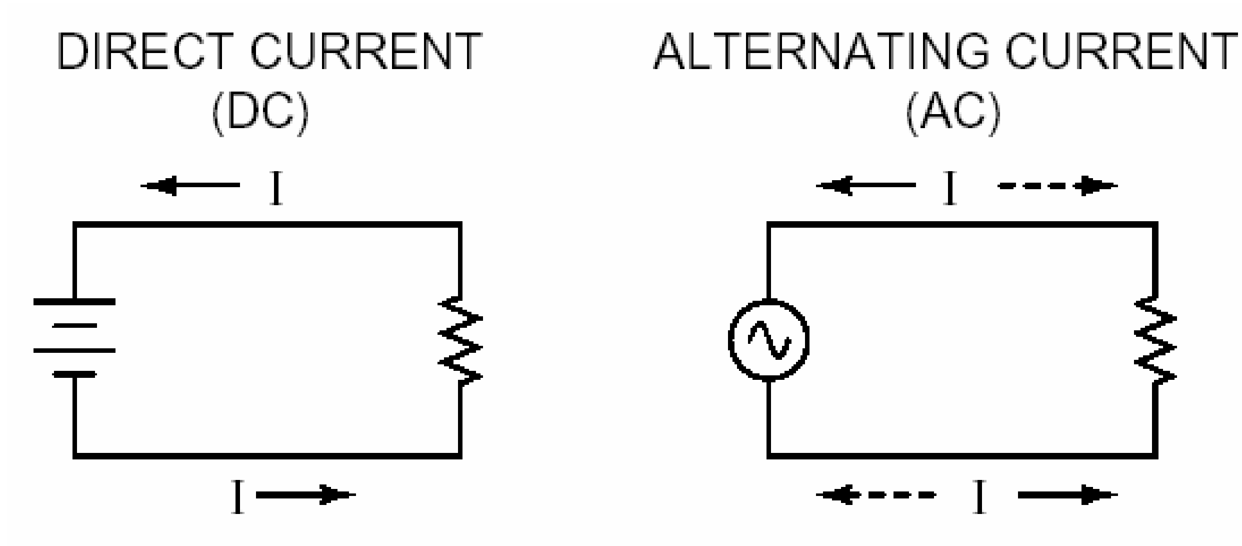
Alternating & Direct Currents

- AC versus DC signals
- AC characterization
- Mathematical tools:
  - Complex number
  - Complex representation of an AC signal
- Resistor in an AC circuit
- Capacitors
- Reactance and Impedance
- RC circuits
- High and low-pass filters

Alternating Current (AC) versus Direct Current (DC)



- With AC it is possible to build electric generators, motors and power distribution systems that are far more effcient than DC.
- AC is used predominately across the world of high power

Alternating Current (AC): waveforms

- AC signal are periodic:

$$S(t+T)=S(t)$$

period

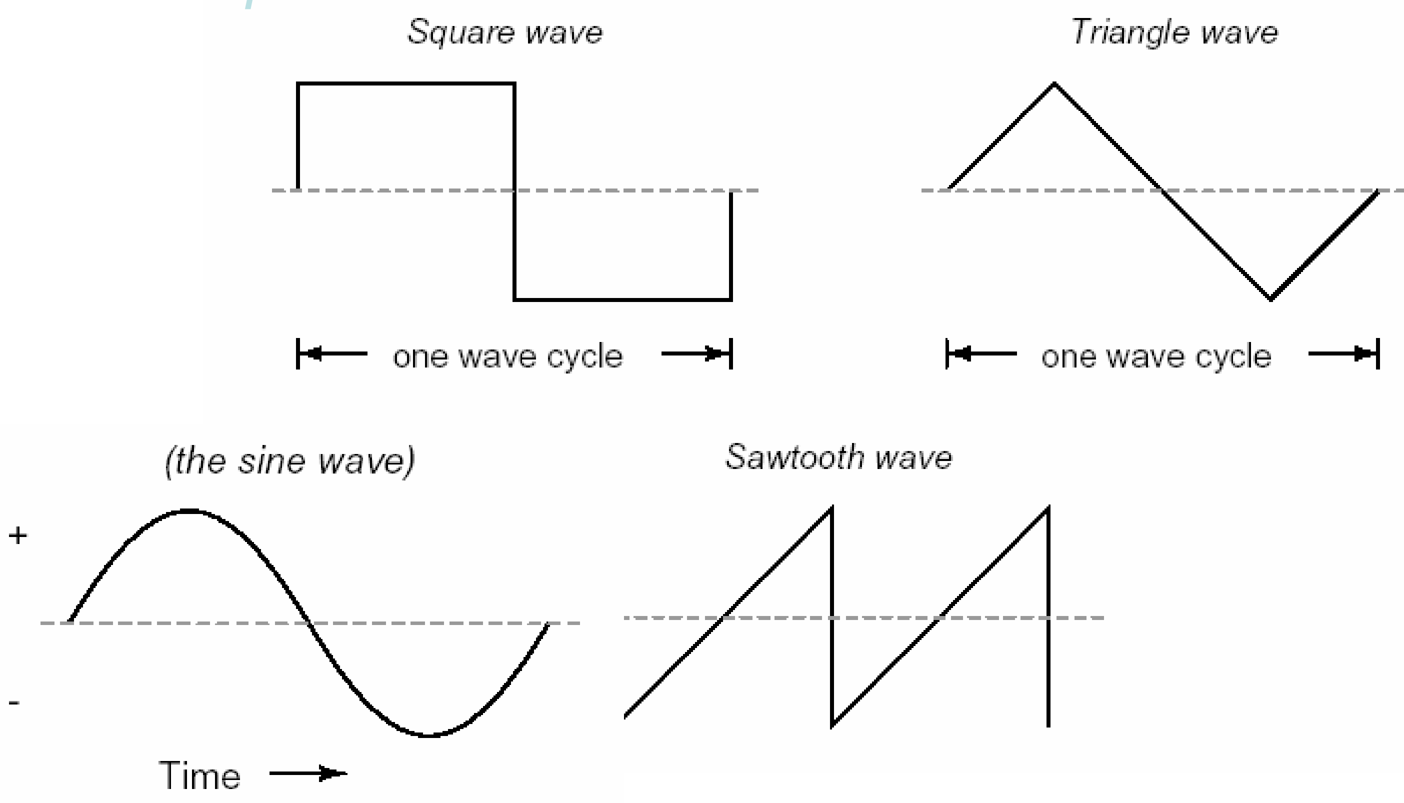
frequency　　pulsation

$$f=\frac{1}{T}=\frac{\omega}{2\pi}$$

<u>*UNITS:*</u>

f: in Hertz (Hz)

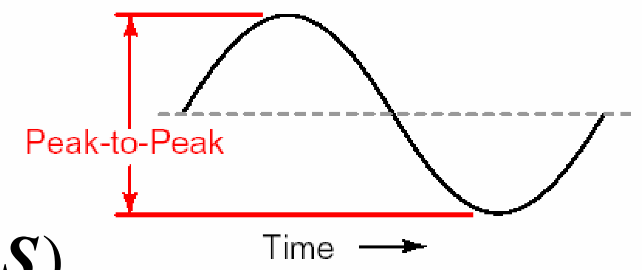
ω: in rad.s^{-1}



Heinrich Rudolf Hertz (1857-1894)

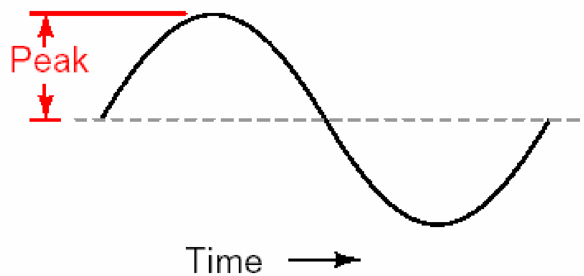
Alternating Current (AC): characterization

- Can an AC waveform be characterized by few parameters?



- Peak-to-peak (PP) $\boldsymbol{PP} = \max(\boldsymbol{S}) - \min(\boldsymbol{S})$

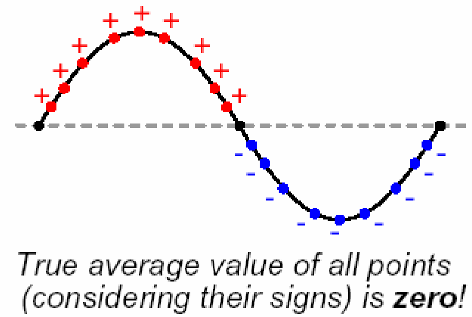
- Peak $P = \max(S)$

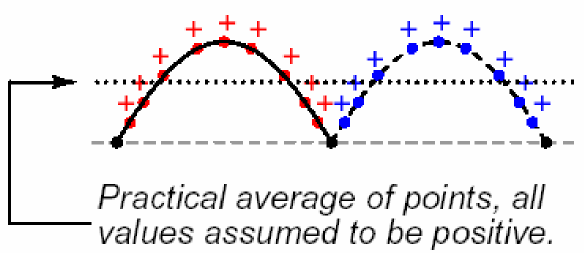


- Average $\langle S \rangle = \frac{1}{T} \int_t^{t+T} S(t)dt$

- Practical Average

$$AVG = \frac{1}{T} \int_t^{t+T} |S(t)| dt$$





- Root-mean-square

$$RMS \equiv \sqrt{\langle S^2 \rangle - \langle S \rangle^2} = \left[\frac{1}{T} \int_t^{t+T} S^2(t)dt \right]^{1/2}$$

where

$$\langle S^n \rangle = \frac{1}{T} \int_t^{t+T} S^n(t)dt$$

Alternating Current (AC): characterization

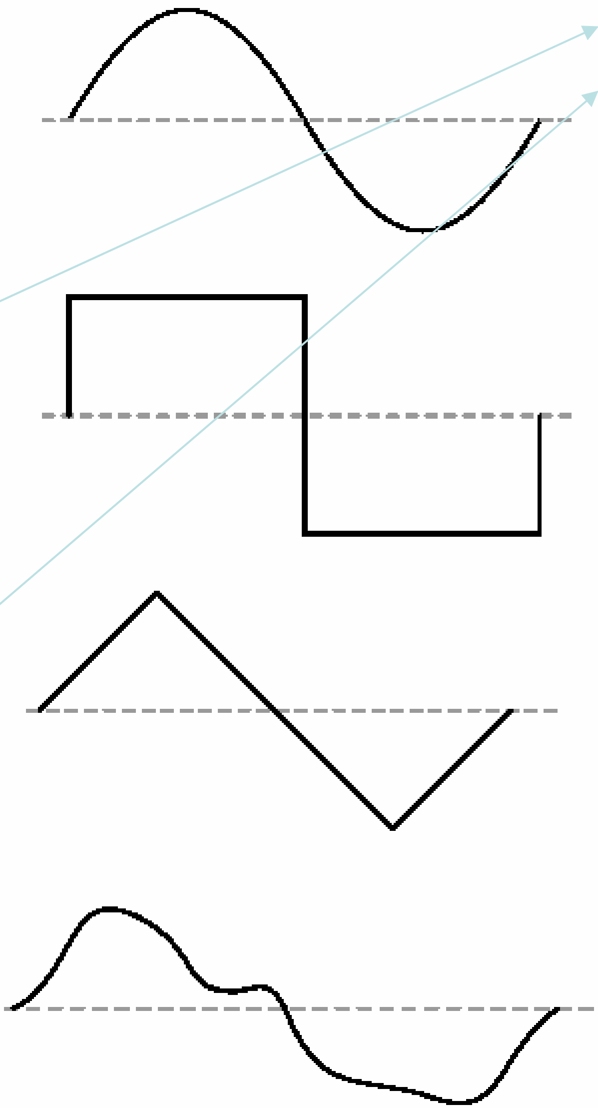
- For some analytical waveform, there exits relation between the different parameters
- Take a sinusoidal waveform with amplitude 1 then

$$(RMS)^2 = \frac{1}{T}\int_t^{t+T} \sin^2(\omega t)dt = \frac{1}{T}\int_0^{2\pi} \frac{1}{\omega}\sin^2(\vartheta)d\vartheta$$

$$\Rightarrow (RMS) = \frac{\sqrt{2}}{2}$$

$$(AVG) = \frac{1}{T}\int_t^{t+T} |\sin(\omega t)|\,dt = \frac{1}{T}\int_0^{2\pi} \frac{1}{\omega}|\sin(\vartheta)|\,d\vartheta$$

$$= 2\frac{1}{2\pi}\int_0^{\pi} \sin(\vartheta)d\vartheta = \frac{2}{\pi}$$



RMS = 0.707 (Peak)
AVG = 0.637 (Peak)
P-P = 2 (Peak)

RMS = Peak
AVG = Peak
P-P = 2 (Peak)

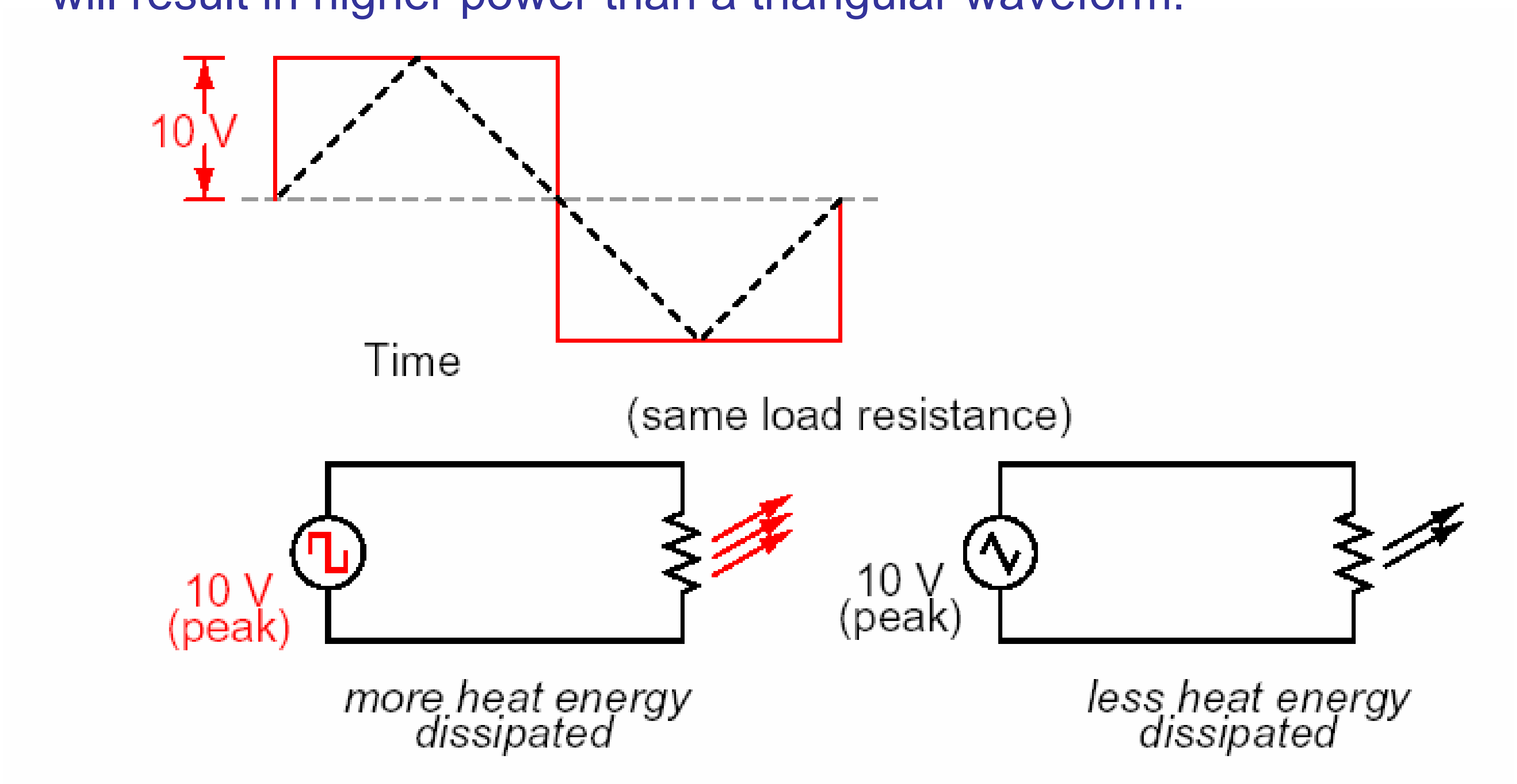
RMS = 0.577 (Peak)
AVG = 0.5 (Peak)
P-P = 2 (Peak)

RMS = ???
AVG = ???
P-P = 2 (Peak)

Alternating Current (AC): characterization

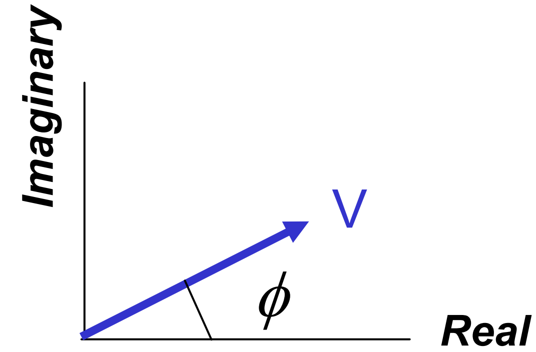
- It matters what waveform is considered

- For instance for the same peak value, a square waveform will result in higher power than a triangular waveform.



Alternating Current (AC): mathematical tools

- In the following we will consider sinusoidal-type waveform (in principle any waveform can be synthesized as a series of sine wave (Fourier)



- We will write (in real notation)

$$S(t) = S_0 \cos(\omega t + \phi)$$

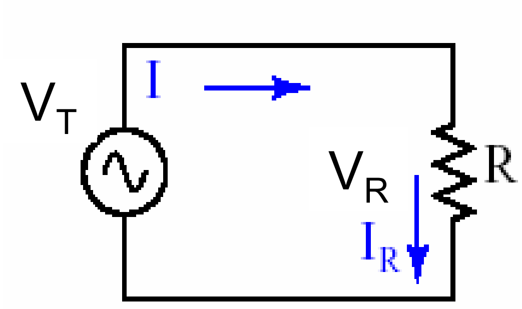
- It often better to use complex notation:

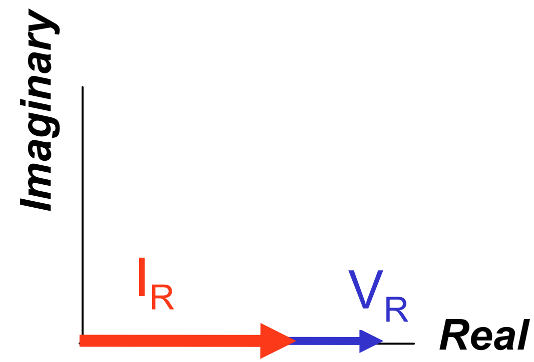
$$S(t) = \Re[S_0 e^{i(\omega t + \phi)}]$$

- And will often do calculation in complex notation and at the end recall that our physical signal is the real part of the complex results

- We can associate a vector in the complex plane to this complex number

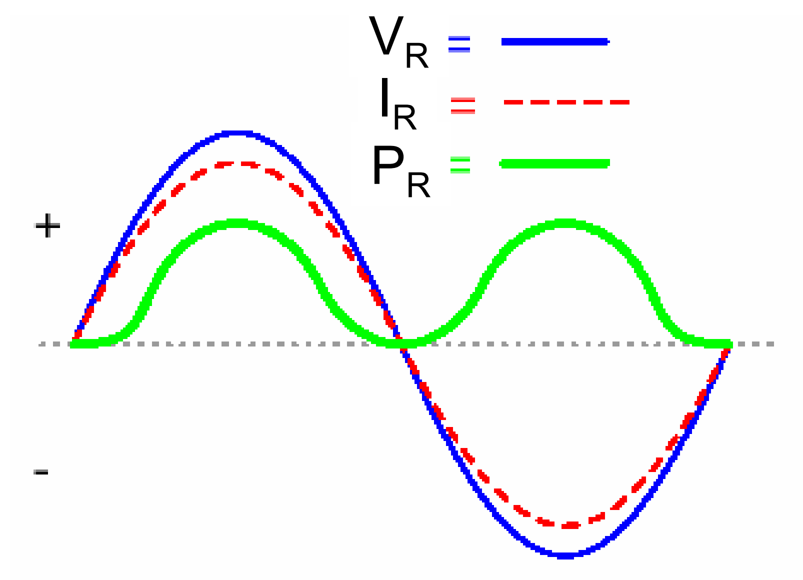
Resistor in an AC circuit



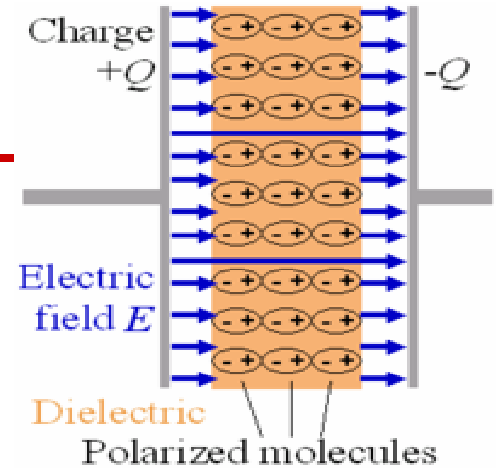


$$V_R = RI_R$$

- *R* is a real number. So in the complex plane, all quantities are along real axis
- Current and Voltage are said to be **in phase**
- When instantaneous value of current is zero
  corresponding instantaneous value of voltage is zero
- Note power > 0 at all time ⇨ resistor always dissipates energy

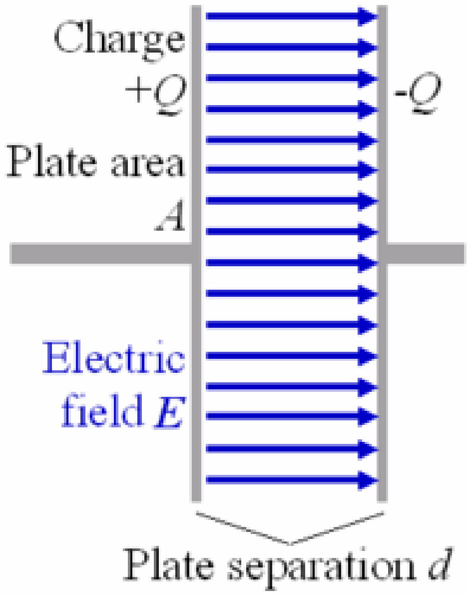


Capacitors: voltage versus current relation



- Current induced by electric displacement:.

$$\vec{J} = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}$$



- Assume a simple model of two plate separated by a small distance. Gauss's law gives:

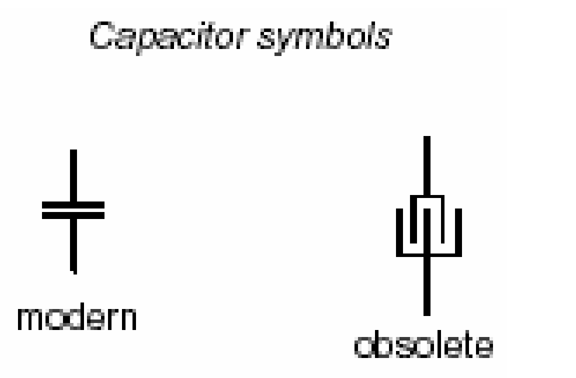
$$\oint \vec{E}.\vec{dS} = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{\varepsilon_0 A}$$

capacitance

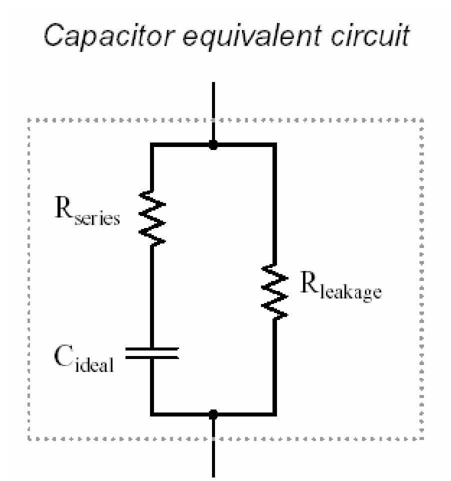
$$\Rightarrow \boxed{V = \frac{QL}{\varepsilon_0 A} \equiv \frac{Q}{C}}$$

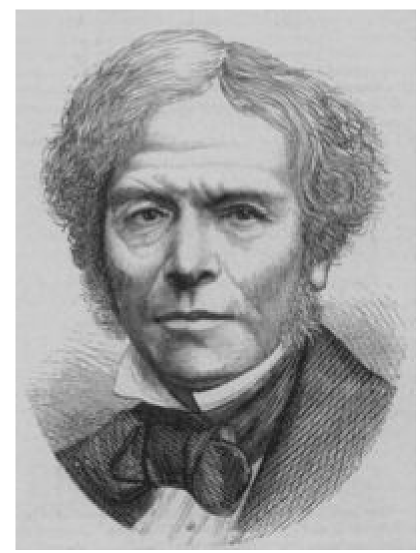
$$\vec{J} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{\partial V}{\partial t} = \frac{Jd}{\varepsilon_0} = \frac{Id}{A\varepsilon_0} = \frac{1}{C} I \Leftrightarrow \boxed{I = C\frac{\partial V}{\partial t}}$$

Capacitors: technical aspects

- Unit for Capacitance is Farad (in honor to Faraday)
- Capacitor symbol:
- 

- Real world capacitors also introduce a resistance (we will ignore this effect)

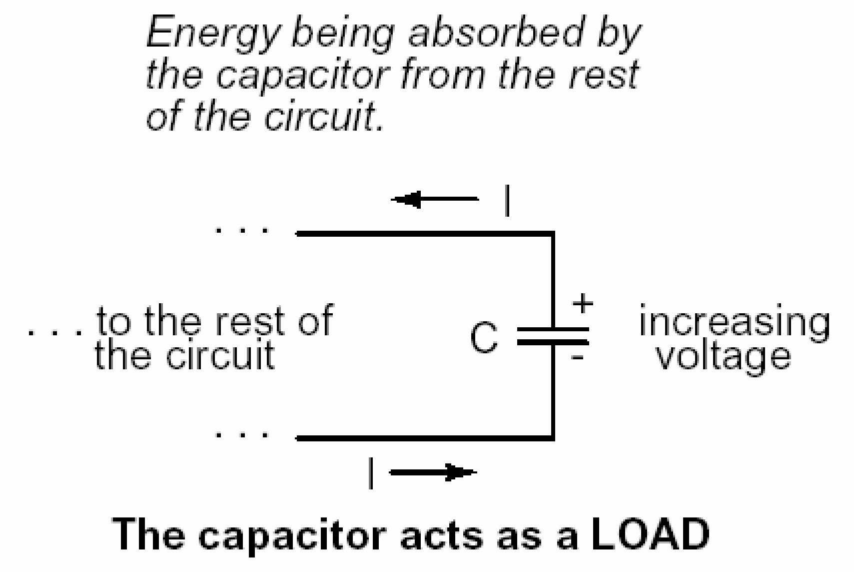


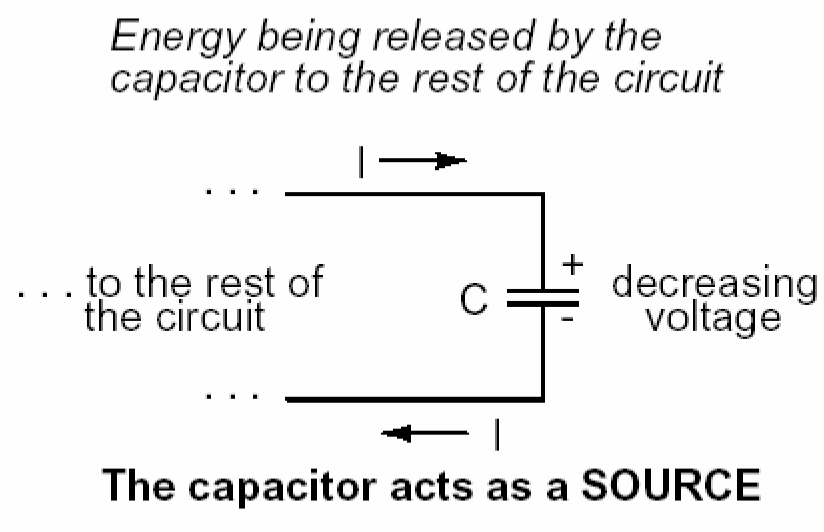


M. Faraday (1791-1867)

Capacitor

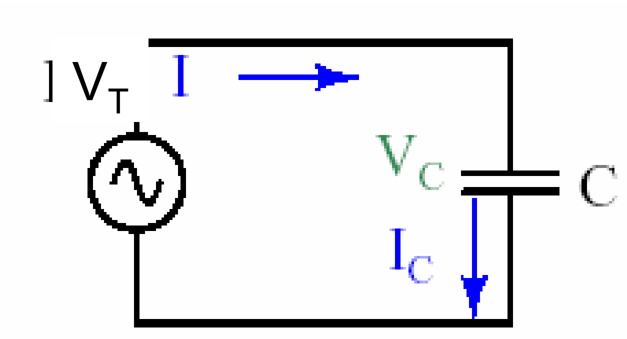
- A capacitor either acts as a load or as a source

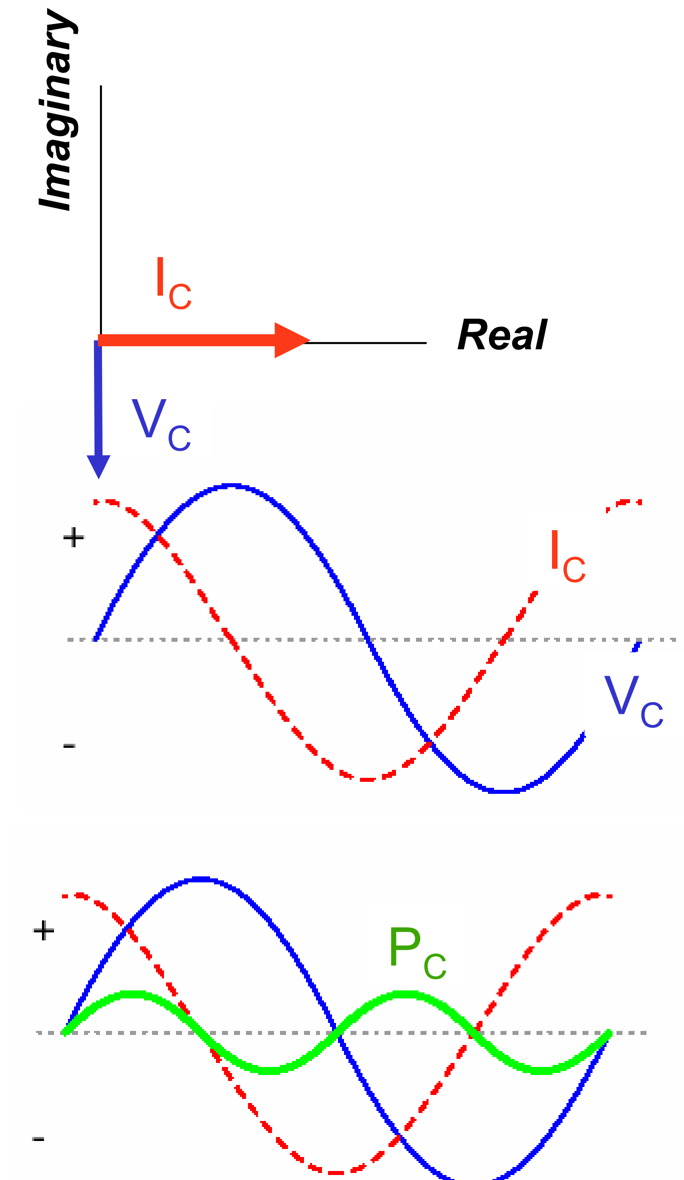




- A capacitor can therefore store energy.

Capacitor in an AC circuits





$$I_C = C\frac{dV_C}{dt} = i\omega C V_C$$

- Capacitors do not behave the same as resistors
- Resistors allow a flow of e- proportional to the voltage drop
- Capacitors oppose change by drawing or supplying current as they charge or discharge.

Reactance and Impedance

- The general linear relation between V and I is of the form

$$\boldsymbol{Z} \equiv \boldsymbol{V} / \boldsymbol{I}$$

Z is called **impedance**.

- For a resistor *Z*=*R* is a **real number**.

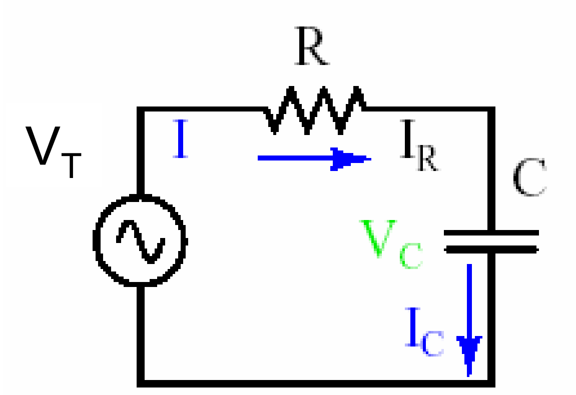
- For a capacitor $Z = \frac{-i}{\omega C}$ is an **imaginary number**

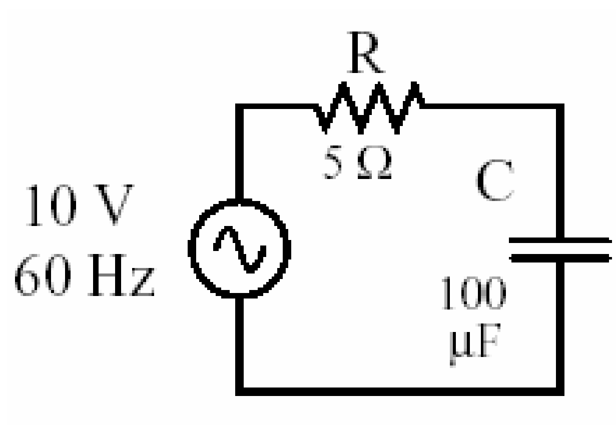
- Generally *Z* will be a **complex number** (if *V* and *I* are written in their complex forms)

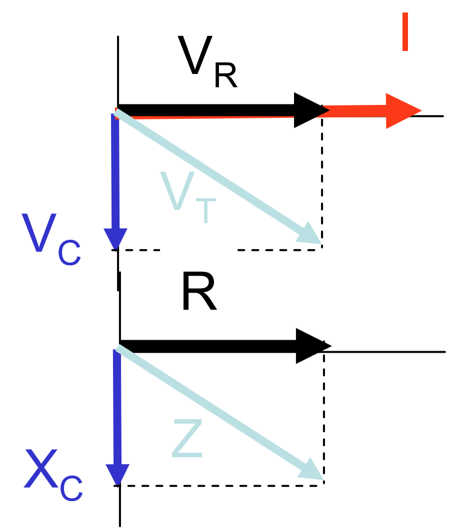
- For instance if a circuit has both capacitor(s) and resistor(s) we expect *Z* to generally be a complex number

- For a capacitor the quantity $X_C = \frac{-1}{\omega C}$ is called reactance and is in Ohm (Ω)

Example: Impedance of a series RC Circuits







- Let's compute the total impedance of the RC circuit:

$$V_T = V_C + V_R = \frac{-i}{\omega C} I + RI = (R - \frac{i}{\omega C}) I$$

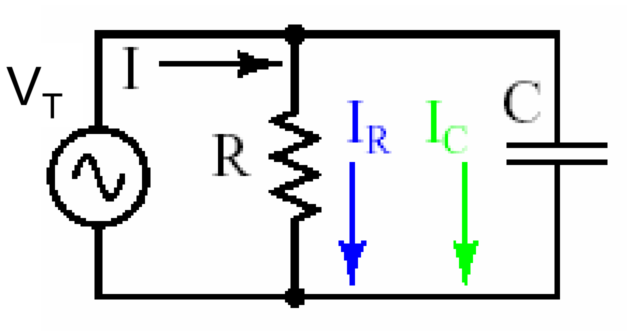
$$\Rightarrow Z = R - \frac{i}{\omega C} = R + iX_C$$

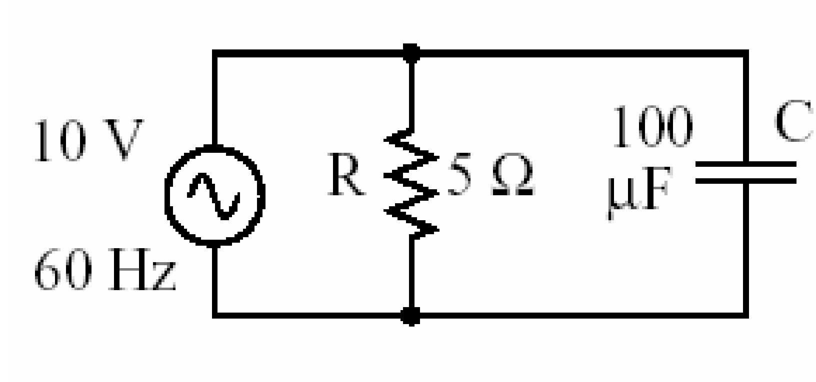
- The impedance can be written as:

$$Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} e^{i\Xi}, \; with \quad \tan \Xi = -\frac{1}{\omega RC}$$

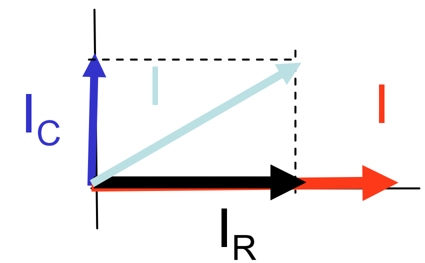
- NA: *Z=5-26.52i* or |Z|=*29.99* and *Ξ=-79.325* degree

Example: Impedance of a parallel RC Circuits



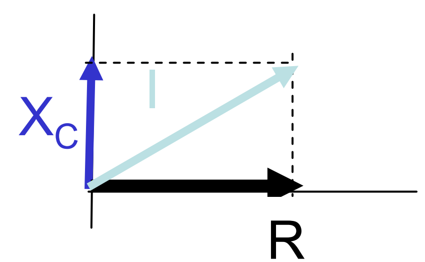


- Let's compute the total impedance of the RC circuit:



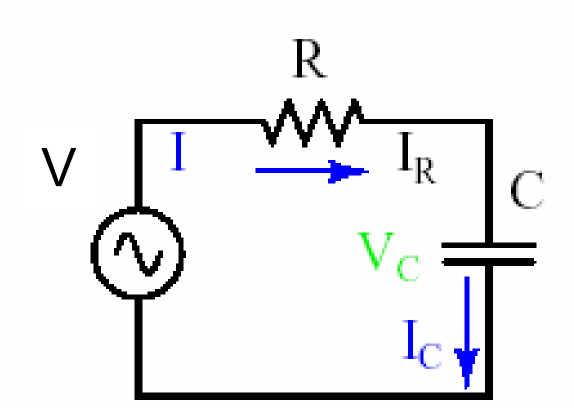
$$I = I_C + I_R = \left(i\omega C + \frac{1}{R}\right)V$$

$$\Rightarrow Z = \left(i\omega C + \frac{1}{R}\right)^{-1} = \frac{1}{\dfrac{1}{X_C} + \dfrac{1}{R}}$$



- NA: *Z=4.83-0.91i* or |Z|=*4.91* and *Ξ=-10.68* degree

General Analysis of an RC series circuits



$$V = V_0 \cos \omega t = \Re\left(V_0 e^{i\omega t}\right)$$

- Let's write the ODE for the current

$$V = V_R + V_C = RI + \int \frac{I}{C} dt$$

$$\Leftrightarrow \frac{dI}{dt} + \frac{1}{RC} I = \frac{1}{R} \frac{dV}{dt}$$

- How do we solve?

Solving the differential equation for the RC series circuit

- Previous equation is of the form:

$$y'(t) + \alpha y(t) = f(t), \quad y(0) = y_0$$

- First find the solution for the homogeneous equation

$$y_h = D\, e^{-\alpha t}$$

- Then find a particular solution of the inhomogeneous equation $y_p(t) = g(t)\, e^{-\alpha t}$

$$f(t) = (g(t)\, e^{-\alpha t})' + \alpha g(t)\, e^{-\alpha t}, \Rightarrow f(t) = g'(t)\, e^{-\alpha t}$$

$$g(t) = \int_0^t f(s) e^{\alpha s}\, ds$$

- The general solution is of the form $y_g = y_h + y_p = e^{-\alpha t}\,(D + \int_0^t f(s) e^{\alpha s}\, ds)$

- So finally we have $\boxed{y(t) = e^{-\alpha t}\,(y_0 + \int_0^t f(s) e^{\alpha s}\, ds)}$

General Analysis of an RC series circuits

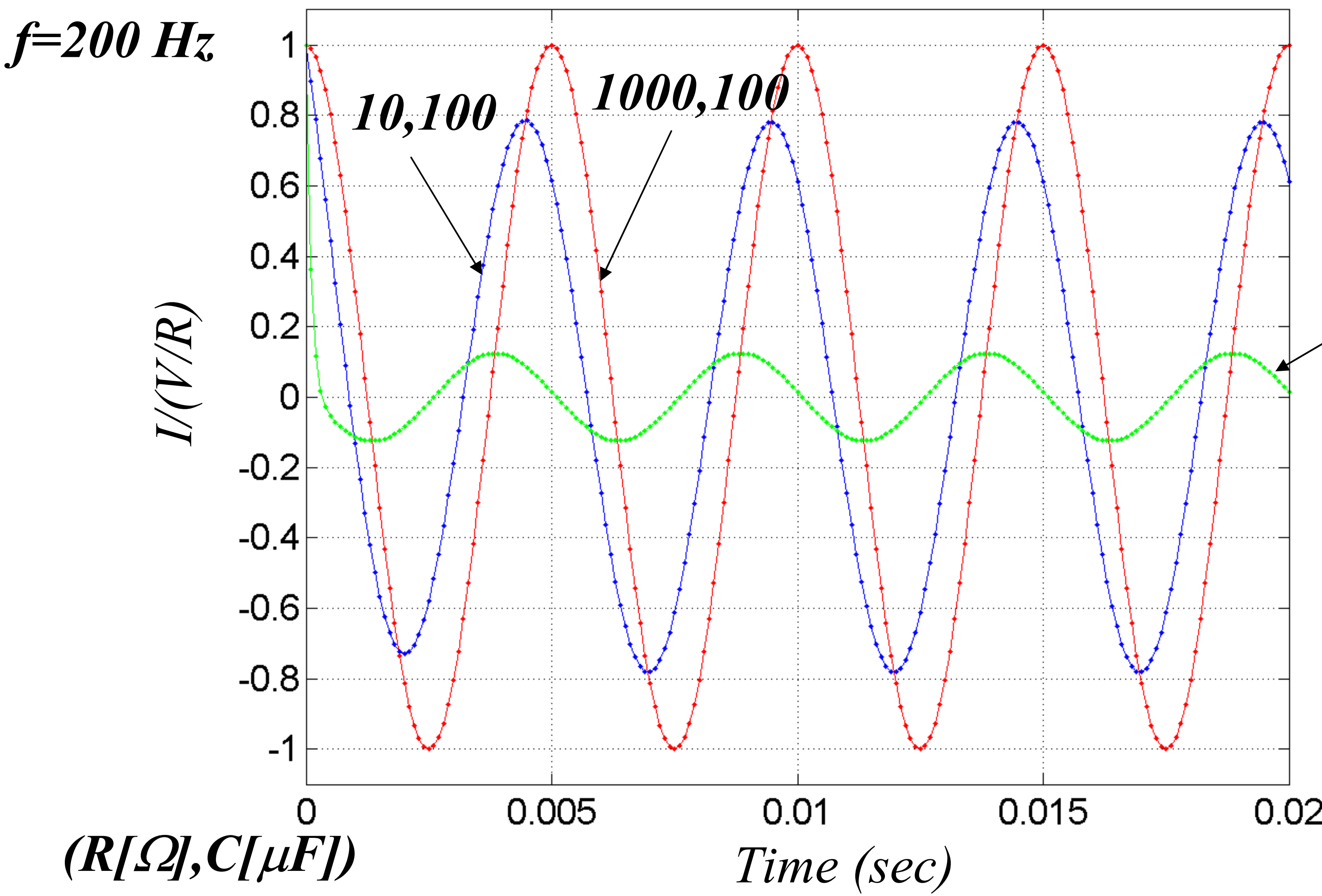
- Applying previous results to RC series circuits gives:

$$I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}} \left(1 + \frac{\omega^2 + \frac{i\omega}{RC}}{\omega^2 + \frac{1}{R^2C^2}} \left(e^{i\omega t + \frac{t}{RC}} - 1 \right) \right)$$

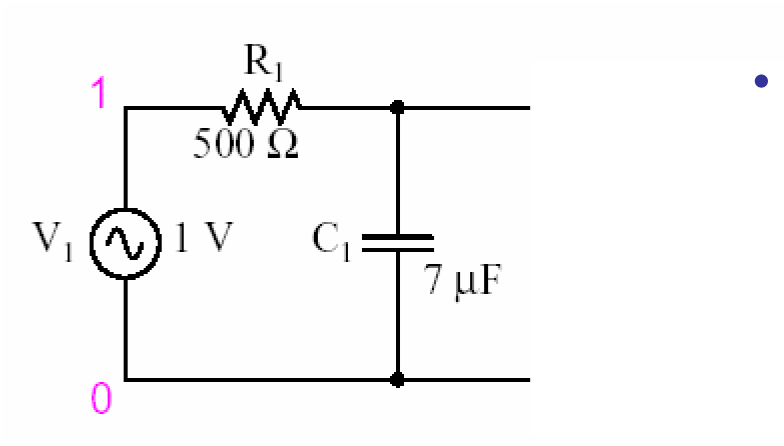
- Or in real notations:

$$I(t) = \frac{V_0}{R} \left(1 - \frac{\omega^2}{\omega^2 + \frac{1}{R^2C^2}} \right) e^{-\frac{t}{RC}} + \frac{V_0}{R} \frac{\omega^2}{\omega^2 + \frac{1}{R^2C^2}} \cos \omega t - \frac{V_0}{R^2C} \frac{\omega}{\omega^2 + \frac{1}{R^2C^2}} \sin \omega t$$

General Analysis of an RC series circuits



RC series circuits as frequency filters: low pass



- The voltage across capacitor is

$$V_c = -\frac{i}{\omega C} I = -\frac{i}{\omega C}\frac{V}{Z}$$

$$\Rightarrow V_c = \frac{1 - iRC\omega}{1 + R^2C^2\omega^2} V$$

- The gain A is defined as: $A = |A| e^{i\Theta}$

$$A = \frac{V_C}{V} = \frac{1 - iRC\omega}{1 + R^2C^2\omega^2}$$

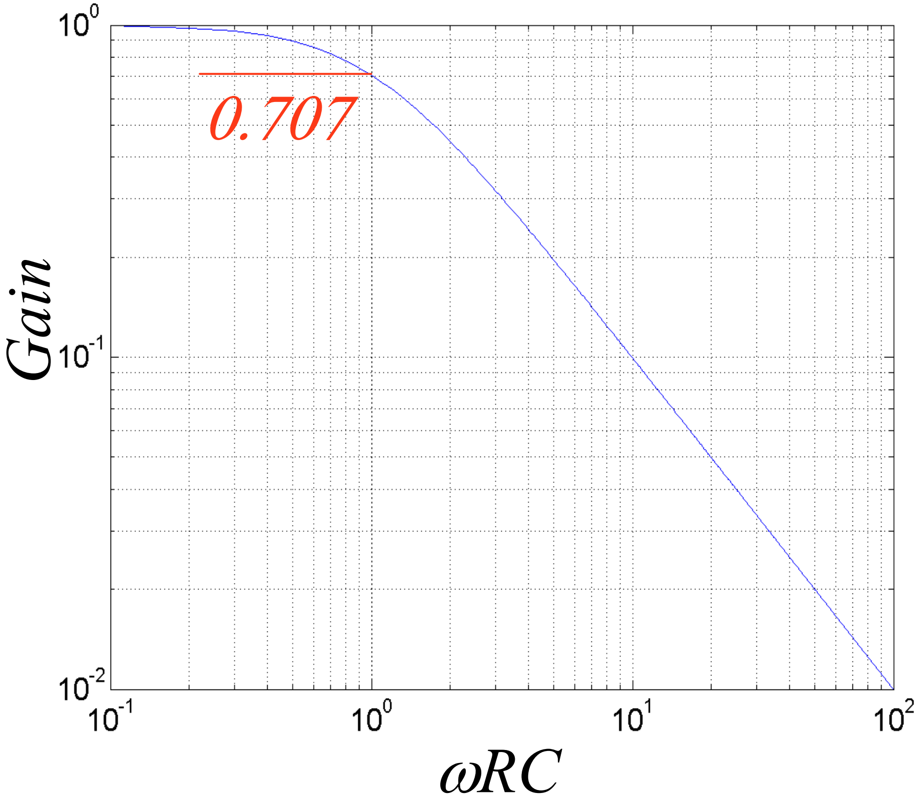
$$\Rightarrow |A| = \frac{1}{\sqrt{1 + R^2C^2\omega^2}}; \Theta = \arctan(-RC\omega)$$

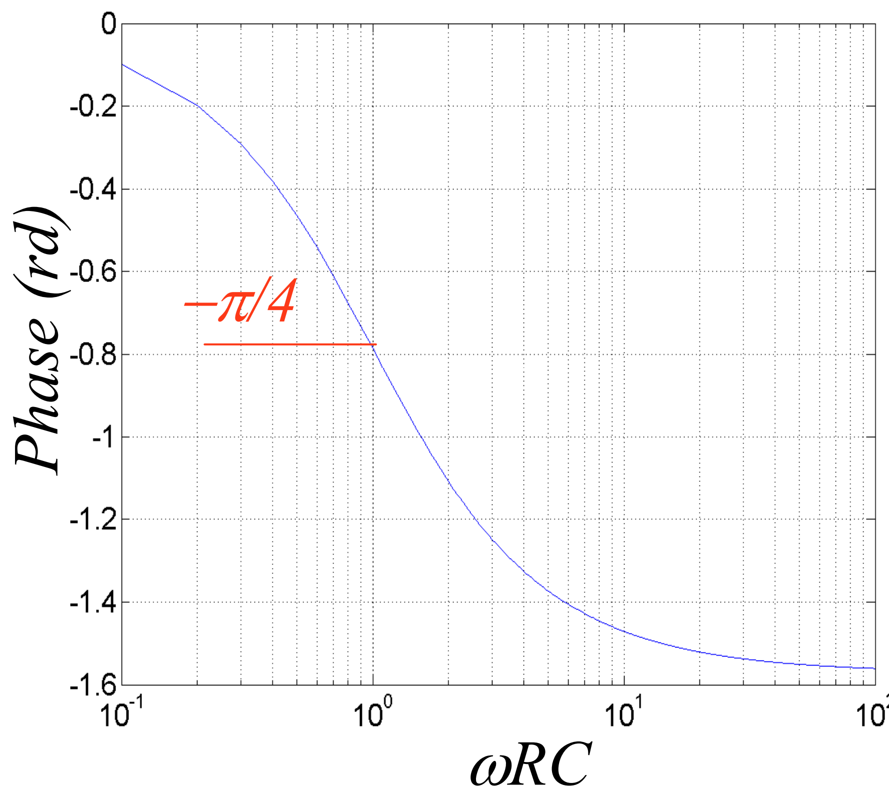
- Note the limits

$$\lim_{\omega >> 1/RC} |A| = 0; \lim_{\omega << 1/RC} |A| = 1$$

$$\lim_{\omega >> 1/RC} \Theta = -90; \lim_{\omega << 1/RC} \Theta = 0$$

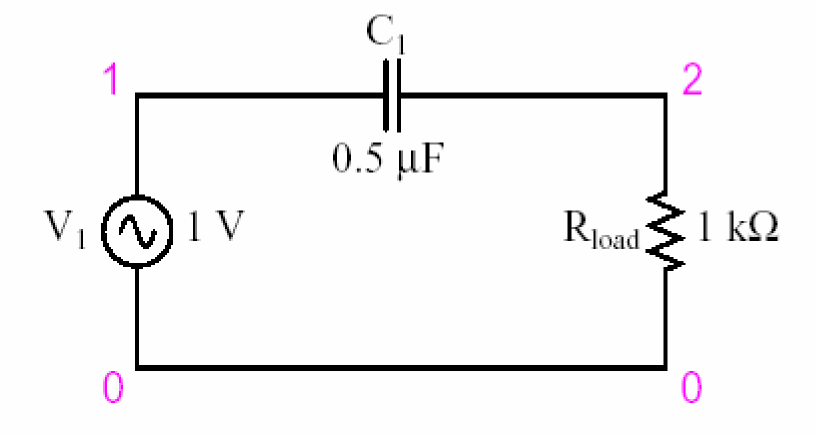
RC series circuits as frequency filters: low pass





- Signal with frequencies **below** 1/RC are **unaltered**,
- Signal with frequency **above** 1/RC are **attenuated**

RC series circuits as frequency filters: high pass



- The voltage across capacitor is

$$V_R = RI = R\frac{V}{Z}$$

$$\Rightarrow V_c = R\frac{1}{R - \frac{i}{C\omega}}V$$

- The gain A is defined as: $A = |A| e^{i\Theta}$

$$A = \frac{V_C}{V} = \frac{R^2C^2\omega^2 + iRC\omega}{1 + R^2C^2\omega^2}$$

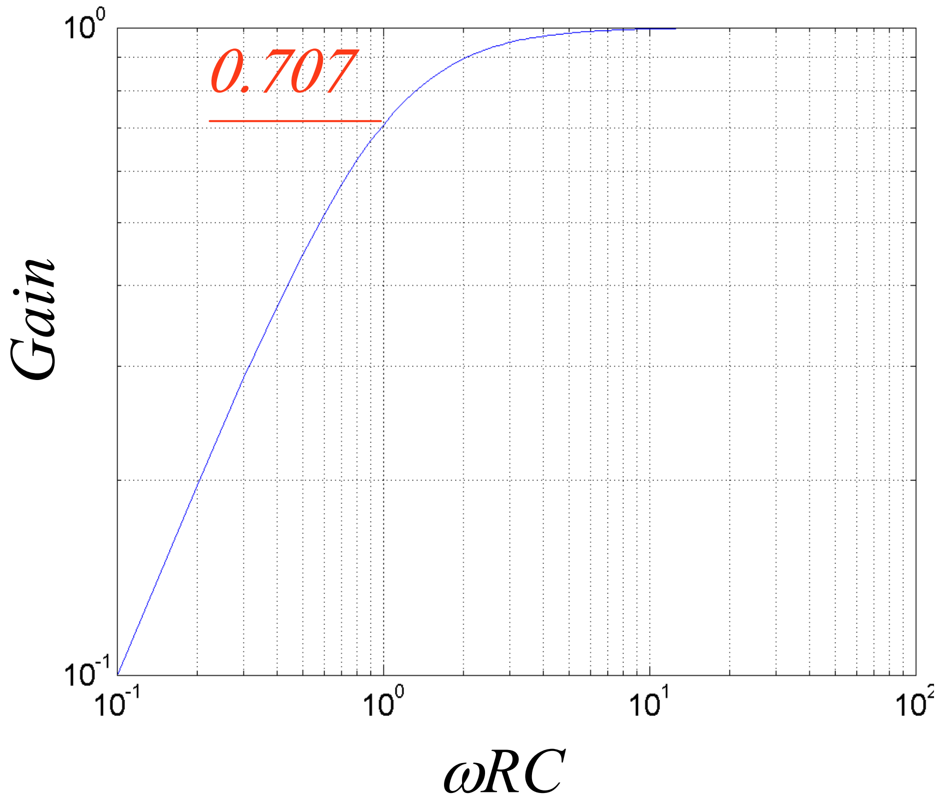
$$\Rightarrow |A| = \frac{RC\omega}{\sqrt{1 + R^2C^2\omega^2}}; \Theta = \arctan\left(\frac{1}{RC\omega}\right)$$

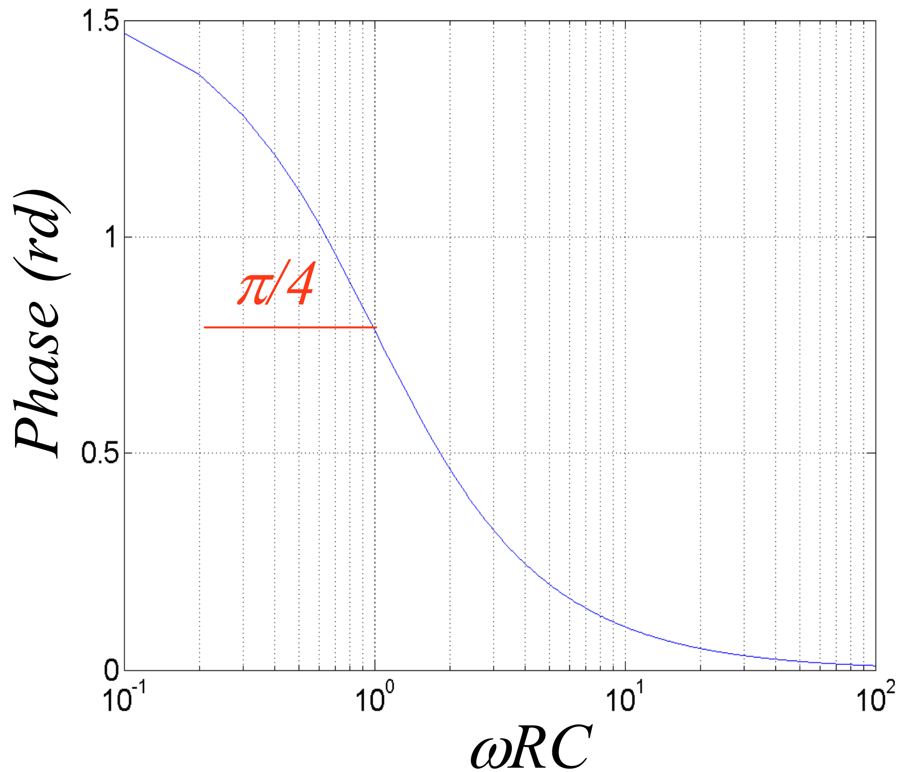
- Note the limits

$$\lim_{\omega >> 1/RC} |A| = 1; \lim_{\omega << 1/RC} |A| = RC\omega$$

$$\lim_{\omega >> 1/RC} \Theta = 0; \lim_{\omega << 1/RC} \Theta = 90$$

RC series circuits as frequency filters: high pass





- Signal with frequencies **above** 1/RC are **unaltered**,
- Signal with frequency **below** 1/RC are **attenuated**