SELF-CONSISTENT MODELS OF TRIAXIAL ELLIPTICAL GALAXIES WITH CENTRAL CUSPS

by

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1. INTRODUCTION

The importance of understanding the structure and the underlining physical laws governing the dynamics and evolution of galactic systems can hardly be overstated. The study of galactic systems is a major tool in comprehending some of the key issues in astrophysics relating to the origin, evolution and structure of the Universe. In addition, galactic dynamics has attracted a great deal of attention due to the wide variety of dynamical phenomena it exhibits. Analysis of the morphology and stability of orbits of individual stars in these galactic systems is essential to developing a viable theory about the behavior of the galaxy as a whole.

Based on observations, we can divide the galaxies according to their shapes into spirals (barred and non-barred), ellipticals and irregulars (with some transitional types). Of particular interest to our study are the elliptical galaxies.

In literature, elliptical galaxies are denoted En, where $n = \frac{(a-c)}{a}$, a and c being the long and the short axis, respectively. Observed ellipticals range from spherical $E0$ to highly flattened $E7$. Observations of elliptical galaxies can give us only a line-of-sight projection of their shape, thus leaving the question of possible axial symmetry unanswered. Until recently, elliptical galaxies were considered to be axisymmetric (oblate) due to their rotation. The change in this common view came as a result of the realization that most elliptical galaxies rotate significantly more slowly than expected for a fluid body with the same flattening (Bertola & Cappaccioli 1975). Most of the resistance against gravitational collapse is due to the random motion of stars and not from the ordered rotation of a galaxy as a whole. These observations gave rise to the question: are elliptical galaxies axisymmetric or fully triaxial? Binney (1978) has proposed that the observed flattenings of elliptical galaxies are mostly due to anisotropic velocity distributions, and noted that the triaxial models are at least as likely as the axisymmetric ones. Shortly after, Schwarzschild (1979) devised a numerical method to construct self-consistent triaxial models from time-averaged orbits. The issue of triaxiality has thus taken a focal point in understanding the dynamics of elliptical galaxies.

In §2 we outline and justify the treatment of elliptical galaxies as collisionless systems, which is the foundation of any work in galactic dynamics. Then, in §3, we describe a number of approaches used for investigating the dynamics of triaxial galaxies. Special attention is paid to scale-free potentials which will play a primary role in our research. A brief summary of the morphology and stability of orbits is also included. In §4 we describe the Hamiltonian formulation of equations of motion, and the methods to construct tori on which regular orbits move; trajectory-following approaches are described in detail, and one such approach which is used extensively in our proposed research will be outlined in §5. §6 briefly summarizes our earlier work, in an attempt to demonstrate the ability of the present author to do original research. An outline of the proposed research in §7 completes the prospectus, with the clear intention to demonstrate that its successful completion should merit a Ph.D. degree to the author.

2. COLLISIONLESS APPROXIMATION

Elliptical galaxies are systems comprised of 10^7-10^{12} stars. Based on the typical size of an elliptical galaxy, one can readily compute the mean free path and the interval between collisions. We define a *collision* to be an encounter between stars when the gravitational effect due to masses involved dominate the gravitational field of the rest of the galaxy. The result obtained, that the collision time is several orders of magnitude larger than the size of the Universe, justifies the treatment of stars in galaxies as point masses and galaxies as collisionless systems (Binney & Tremaine 1987). Similarly, it can be calculated that the gravitational effect of the nearest star is negligible compared to the effects of the rest of the mass in the galaxy. This allows us to use a continuous mass distribution to obtain a useful approximation to a galaxy's gravitational field.

2.1. The Collisionless Boltzmann Equation

The dynamics of galaxies approximated as collisionless systems of continuous mass distribution is described through a six dimensional (6-D) phase space distribution function (DF) $f(\vec{x},\vec{v},t)$, satisfying the collisionless Boltzmann equation (CBE)

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\bigtriangledown} f - \vec{\bigtriangledown}\Phi\cdot\frac{\partial f}{\partial \vec{v}} = 0, \tag{1}$$

with Φ being the gravitational potential of the system. In order for the DF to be physically meaningful, it must be non-negative. Equilibrium models have $\frac{\partial f}{\partial t} = 0$. Integration of the DF over the velocity space yields the *luminous mass density* $\rho_l(\vec{x})$ of the system:

$$\rho_l(\vec{x},t) = \int\int\int f(\vec{x},\vec{v},t)d^3\vec{v}. \tag{2}$$

The luminous mass density is a product of the *mass-to-light ratio* $(\mathcal{M}/\mathcal{L})(\vec{x})$ and the *luminosity density* $j(\vec{x})$. The *mass density* ρ gives rise to the gravitational potential $\Phi(\vec{x})$. The two quantities are related through Poisson's equation

$$\bigtriangledown^2\Phi = 4\pi G\rho, \tag{3}$$

where G is the gravitational constant. The system is *self-gravitating* when only the luminous mass density is responsible for the entire gravitational potential, i.e. $\rho(\vec{x}) \equiv \rho_l(\vec{x})$. On the other hand, *dark matter*, such as a dark galactic halo or a central black hole, can contribute to the expression for the mass density $\rho(\vec{x})$. In order to have a *dynamical model* for a self-gravitating galactic system, we must solve equations (1)-(3) simultaneously. If they can be solved, we refer to the model as *self-consistent*.

In modeling the dynamics of elliptical galaxies, one can adopt one of the following two approaches:

1. "from ρ to f", where the mass density ρ is specified and (1)-(3) inverted and solved for f.

2. "from f to ρ", where the DF is specified and (1)-(3) solved directly for ρ.

In either of these approaches, it is useful to employ Jeans' theorem, which tells us how the DF and the phase-space coordinates are related.

2.2. Integrals of Motion and Jeans' Theorem

Any *isolating integral of motion* $I(\vec{x}, \vec{v})$, a quantity conserved on orbits in a given potential, satisfies the CBE (1). The isolating integrals of motion and the DF are related through Jeans' theorem which states that the DF depends on phase space coordinates $(\vec{x}, \vec{v})$ only through three isolating integrals of motion, $f(I_1, I_2, I_3)$ (Jeans 1915; Lynden-Bell 1962). If the system admits fewer integrals of motion than degrees of freedom (dof), the orbits are *stochastic*; this means that each orbit will chaotically fill the portion of the phase space delimited by the curve(s) of constant integral(s) of motion that it obeys. If, on the other hand, the number of isolating integrals of motion is equal to the number of dof, the orbits are *regular*.

It is useful to apply Jeans' theorem to the construction of self-consistent models because the DF is not a function of 6 phase-space coordinates, but only (at most) 3 integrals of motion.

3. DYNAMICS OF TRIAXIAL GALAXIES

Triaxial galaxies have three planes of reflection symmetry, but there are no symmetry axes. No component of the angular momentum is conserved. This absence of symmetries makes triaxial galaxies a 3-dof problem.

Most triaxial galaxies admit only one exact isolating integral of motion, the orbital energy E (defined as the value of the Hamiltonian (10) describing the orbit in the phase space). However, the numerical models show that a fair portion of orbits in triaxial systems are regular (Schwarzschild 1979, 1982, 1993; Miralda-Escudé & Schwarzschild 1989; Merritt & Valluri 1998; Valluri & Merritt 1998). These orbits admit two isolating integrals of motion I_2, I_3 in addition to the orbital energy E, and the DF is $f(E, I_1, I_2)$. However, since these isolating integrals of motion are not explicitly known, we cannot directly apply Jeans' theorem to construct a physical DF. To gain insight into the dynamics of triaxial galaxies, two approaches have been employed:

1. *Numerical*

a. *Schwarzschild's method* (Schwarzschild 1979) divides the model into cells of a 3-D sphere. Based on the amount of time it spends in each of the cells i, the *orbital density* ρ_{ij} for each orbit is computed. Now, we seek the set of non-negative weights w_i for each of the orbits, such that the weighted sum of all the orbital densities of the model will reproduce the starting density distribution of the model ρ_i in each of the i cells. That is,

$$\rho_i = \sum_{j=1}^{N} w_j \rho_{ij}, \tag{4}$$

where N is the number of orbits and the normalized orbital densities are given by

$$1 = \sum_{i=1}^{M} \rho_i, \tag{5}$$

with M being the number of cells in a 3-D sphere.

(4)-(5) constitute an optimization problem and can be solved in several ways, the most popular of which are linear programming methods (Schwarzschild 1979), non-negative least squares (Pfenniger 1984) and Lucy iteration (Statler 1987).

The advantage of scale-free potentials is that one does not need to consider a 3-D model, but only a 2-D reference sphere, and in the case of triaxial symmetry, only an octant of it (Levison & Richstone 1987). This is a consequence of the fact that each orbit in the scale-free potentials is representative of the whole ensemble of orbits; for each point on an integrated orbit at a fixed orbital energy E_0, there exists an analogous point on a similar orbit (at some other energy E_1) which lies on a reference sphere (Richstone 1980; Schwarzschild 1993). The solutions to (4)-(5) are highly non-unique (Hunter 1995). Mathematically, the reason for this is that the problem (4)-(5) is underspecified. Physically, the orbits that comprise the distribution function can be combined in a number of ways to yield the same density function.

Schwarzschild's method is the standard approach to verifying whether the model is self-consistent or not.

b. *N-body simulations* model galactic systems by numerically computing the gravitational interactions of the order of 10^6 point masses. This approach can give us a valuable insight into equilibrium models, instabilities and the secular evolution of the models.

Some recent N-body simulations of galaxy formation which include a dissipative component show an evolution toward axisymmetry (Udry 1993; Dubinski 1994; Barnes & Hernquist 1996). A central point mass, modeling a central black hole, has a similar effect (Norman et al. 1985; Merritt & Quinlan 1998). These results seem to imply that the triaxiality of galactic systems is not as prevalent as once thought, and that it may be a transient phenomenon.

However compelling the results of these N-body simulations may be, we cannot embrace them as the last word on the issue of triaxiality of elliptical galaxies, because of the number of approximations and simplifications introduced in their implementation; these simulations also deal with up to 10^6 stars, which is several orders of magnitude smaller than a realistic galactic system. Instead, we use N-body simulations as auxiliary methods in instances when other treatments become impractical for technical reasons.

2. *Semi-analytical* approaches study special cases of potentials for which sufficient simplifications are introduced allowing some analytical treatment.

a. *Separable potentials* owe their name to the property that the Hamilton-Jacobi equations of motion separate in confocal ellipsoidal coordinates (de Zeeuw 1985). The equations of 3-D orbital motion are derivable analytically, and it is a combination of three 1-D motions (either oscillation or rotation) in each of the coordinates of the ellipsoidal coordinate system. The exact isolating integrals of motion are analytically known, so Jeans' theorem applies. Direct derivation of the DF $f(E, I_2, I_3)$ corresponding to the given triaxial density $\rho(\vec{x})$ is difficult and cumbersome, yet doable (Dejonghe & Laurent 1991; Hunter & de Zeeuw 1992). These potentials have homogeneous cores with non-singular density profiles.

The analysis of separable potentials was shown to be quite worth the effort, as one is able to analytically derive the four major orbital families found in numerical models (Schwarzschild 1979). However, as we will stress later, density profiles with homogeneous cores are not a good physical model of a realistic galaxy.

b. *Scale-free potentials* give rise to orbits whose properties simply scale with radius.

Our research will focus on scale-free potentials, so we will describe their properties in some detail.

3.1. Scale-free Potentials

Recent ground-based (Moller et al. 1995) and space telescope (Crane et al. 1993; Lauer et al. 1995) observations show that the elliptical galaxies essentially never have constant-density cores; the stellar surface brightness continues to rise into the smallest observable radius. This justifies our use of *power-law* potentials (subset of which are the scale-free ones), as opposed to the ones with a constant-density core. It is very important to realize that the dynamics in these two models is substantially different.

The usual form of a density profile of a power-law model is

$$\rho = \rho_0 m^{-\gamma}(m_0 + m)^{-\alpha+\gamma}, \tag{6}$$

where

$$m(x,y,z) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^{1/2}, \quad a \geq b \geq c. \tag{7}$$

a, b and c are lengths of the long, intermediate and short axes of the galaxy, respectively. m_0 is the core radius. $-\gamma$ is the power-law behavior of the density profile near the center of the galaxy, and $-\alpha$ is the power-law behavior in the outer parts. The power-law model is said to be *scale-free* when $m_0 = 0$.

In scale-free potentials all orbits are similar when scaled. This reduces the parameter space to be surveyed and simplifies the construction of self-consistent equilibria, which is the main reason for popularity of these models. Another reason is that scale-free potentials may feature singular central density profiles (*cusps*). Recent Hubble Space Telescope observations have shown that cusps are a common feature of early-type elliptical galaxies, and that some appear even to have nearly pure power-law profiles (Lauer et al. 1995).

The scale-free triaxial potentials we will deal with arise from the ellipsoidal equidensity mass distribution

$$\rho = \frac{\rho_0}{m^\beta}, \tag{8}$$

which, by Poisson's equation (3), yields potential of the form

$$\Phi = \frac{F(\theta, \phi)}{m^{\beta-2}}. \tag{9}$$

θ is a colatitude angle measured from the z-axes, ϕ is an azimuthal angle in spherical coordinates and F is some function of these.

It is apparent from (8) that scale-free potentials have singular central densities for the non-negative values of β. They are classified as *weak* for $\beta < 2$ and *strong* cusps for $\beta \geq 2$. These singular central profiles of scale-free potentials are in contrast to the non-singular homogeneous cores of separable models, which causes the stellar dynamics in two types of potentials to be different.

The motion in the scale-free potentials is not separable. The only known isolating integral of motion is E, although many orbits obey two additional isolating integrals and are regular.

The presence of a black hole as a central point mass, for which there is a great deal of evidence for many galaxies (Ford et al. 1994; Miyoshi et al. 1995), destroys the scale-free nature of the potential.

3.2. Stellar Orbits in Triaxial Galaxy Models

A careful treatment of irregular orbits is necessary if we are to include them into our numerical models. If a star on its orbit makes only a few rotations around the galactic center during the Hubble time (the age of the Universe), it may behave like a regular orbit for long integration times (*weakly stochastic*). In that case, we essentially treat these stochastic orbits as regular. An example of these orbits are the ones in the outer parts of the galactic halo. If, on the other hand, the star makes many rotations around the galactic center during the Hubble time, we can consider this orbit fully *ergodic* (it fills all of phase space uniformly). The orbital density of these fully ergodic stochastic orbits is generally rounder than the model and are not very useful for reconstructing the desired mass distribution. For this reason, stochastic orbits are sometimes completely deleted from the search for the self-consistent solutions (Merritt 1997). An example of this kind of orbits are the ones of stars in the central galactic bulge.

The extent to which regular and stochastic orbits can be *mixed* is one of the key issues in building the self-consistent triaxial models and should be very carefully addressed. Numerical models have shown that the regular stellar orbits alone are sufficient to build self-consistent models only for nearly axisymmetric or nearly spherical shapes (Schwarzschild 1993; Merritt 1997; Merritt & Valluri 1999; Valluri & Merritt 1998). As the models become flatter and more strongly triaxial, a large portion of boxes, the only regular family of orbits that produces enough density near the center and the long axis of the model, are replaced with resonant orbits which do not have the desirable orbital density for reproducing these flattened models (Schwarzschild 1993). This is why building self-consistent models necessitates the inclusion of irregular orbits.

Orbits are building blocks of any galactic model. Numerical models (Schwarzschild's method, numerical integration of power law and scale-free potentials) and analytical studies of separable potentials (de Zeeuw 1985) have shown that triaxial models feature four major families of regular orbits:

1. *Boxes* are confined to the box-like region around the galactic center. They don't have a general sense of rotation, and their orbit-averaged angular momentum is zero (Figure 1.a).

2. *Short-axis tubes* are confined to a tube-like region around short axis of the galaxy. They have a definite sense of rotation about the short axis (Figure 1.b).

3. *Inner long-axis tubes* move in the region around the long axis tube, around which they have a definite sense of rotation (Figure 1.c).

4. *Outer long-axis tubes* also move in the region around the long axis tube, around which they have a definite sense of rotation (Figure 1.d).

Another important family of orbits arises in non-separable potentials when there is a resonance between the oscillation frequencies in two or three coordinates. The bifurcations of box and tube orbits produce *boxlets* and *tubelets*, respectively. These *resonant* families generally bifurcate in stable/unstable pairs (Hunter et al. 1998; Terzić 1998), which points to the existence of local, confined chaos. Since only non-integrable potentials feature chaotic orbits, it is clear that these orbits cannot exist in the integrable potentials, such as separable ones. This is why the separable models are not a good physical model of a realistic galaxy.

In many models, the resonant orbits occupy a significant portion of the phase space. Their phase-space signatures are small islands which lie on a single curve - a porous torus or *cantorus*. In the vicinity of these cantori there are stochastic orbits which can "diffuse" through them. This is called *Arnold diffusion*. Generally, the resonant orbits are infinitely dense in the phase space, but they still compose a set of measure zero. It is important to note that the boxlets and box orbits that they bifurcate from are topologically not the same.

FIGURE 1. Orbits in non-rotating triaxial potentials. Clockwise from the top left:(**a**) box orbit; (**b**) short-axis tube orbit; (**c**) inner long-axis tube orbit; (**d**) outer long-axis tube orbit (Binney & Tremaine 1987).

FIGURE 2. Boxlet orbits in cuspy triaxial potentials. The ratios of frequencies in x and y are given, along with their descriptive names. Left: centrophobic (stable). Right: centrophilic (unstable) (Miralda-Escudé & Schwarzschild 1989).

4. TORUS CONSTRUCTION

Galaxies are non-dissipative Hamiltonian systems. The equations of motion of stars in a given galactic potential are most elegantly and compactly expressed through Hamilton's equations; in Cartesian coordinates, where generalized coordinates are $\vec{q} = \vec{x}$ and their conjugated momenta $\vec{p} = \vec{v}$, they are given by

$$H(\vec{q}, \vec{p}) = \sum_{i=1}^{N} \frac{p_i^2}{2} + \Phi(\vec{q}), \tag{10}$$

$$\dot{q}_i = \frac{\partial H(\vec{q}, \vec{p})}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(\vec{q}, \vec{p})}{\partial q_i}, \quad i = 1, ..., N, \tag{11}$$

where N is the number of dof. (In order to better exploit the symmetries of the model, some other choice of coordinate system may be more suitable for initial formulation).

Regular motion in N dof admits at least N isolating integrals of motion. It can be viewed as motion on a N-D torus which is described by the *action-angle* coordinates $(\vec{J}, \vec{\theta})$. For any system, a canonical transformation $(\vec{x}, \vec{v}) \mapsto (\vec{J}, \vec{\theta})$ can be obtained, such that on each orbit

$$\dot{J}_i = 0, \quad \dot{\theta}_i = \omega_i, \quad i = 1, ..., N. \tag{12}$$

For the new set of phase space coordinates, Hamilton's equations simplify, as Hamiltonian is now only a function of actions

$$H = H(\vec{J}), \quad \omega_i = \dot{\theta}_i = \frac{\partial H}{\partial J_i}, \quad i = 1, ..., N. \tag{13}$$

The actions, J_i, are constant on each orbit. The angle coordinates, θ_i, are linear functions of time (equation (17)). There are as many fundamental frequencies as there are dof. Generally, the ratios of these fundamental frequencies are irrational, in which case the orbits completely cover the surface of the torus after a sufficiently long time interval. In some cases, *resonance* occurs; that is, the ratio of fundamental frequencies is a rational number, and their relationship may be expressed as $\vec{n} \cdot \vec{\omega} = 0$, where $\vec{n}$ is an integer vector. Every resonance reduces the dimensionality of an orbit at which it occurs by 1. In 2 dof, that means that the orbit will have the dimensionality of 1 and it will close on itself (e.g. 2-D harmonic oscillator); in 3 dof, the orbit will lie on a 2-D membrane, but will not close on itself (for closure, 2 independent resonances are necessary). In general, for an orbit in N dof to close on itself and be periodic, it is necessary to have $N-1$ independent resonances (Merritt 1999).

The desirability of expressing the equations of motion in terms of actions and angles is apparent. Unfortunately, only in special cases is it possible to find analytically a canonical transformation which will transform the Cartesian coordinates $(\vec{x}, \vec{v})$ into action-angles $(\vec{J}, \vec{\theta})$. Generally, our only hope is to be able to numerically relate the Cartesian and action-angle variables. There are two approaches to this problem of torus construction:

1. *Iterative*

a. *Perturbative methods* (Gerhard & Saha 1991) exploit our knowledge of some integrable potentials for which the full set of actions and angles is known analytically to give a perturbation expansion in terms of a small perturbation parameter ϵ, with the Hamiltonian being

$$H = H_0(\vec{J}) + \epsilon H_1(\vec{J}, \vec{\theta}), \tag{14}$$

where $H_1(\vec{J}, \vec{\theta})$ is a perturbation potential. The problem is first transformed into the action-angle coordinates of the unperturbed problem. Then, the relation between the action-angles of the perturbed and the unperturbed problem are established through the perturbation expansion in terms of the perturbation parameter ϵ. One of the most elegant and powerful ways to derive this conversion from one coordinate system to another is through Lie transforms (see §6.2).

b. *Non-perturbative* iterative methods attempt to relate Cartesian variables to action-angles in a more direct fashion.

Ratcliff and collaborators (Ratcliff et al. 1984) have formulated the dependency of Cartesian coordinates on angles as a set of non-linear equations. They realized that the equations of motion of a 3-D orbit

$$\ddot{x} = -\frac{\partial \Phi}{\partial x}, \quad \ddot{y} = -\frac{\partial \Phi}{\partial y}, \quad \ddot{z} = -\frac{\partial \Phi}{\partial z}, \tag{15}$$

can be written in the form

$$\begin{aligned}
\left(\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} + \omega_3 \frac{\partial}{\partial \theta_2}\right)^2 x &= -\frac{\partial \Phi}{\partial x}, \\
\left(\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} + \omega_3 \frac{\partial}{\partial \theta_2}\right)^2 y &= -\frac{\partial \Phi}{\partial y}, \\
\left(\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} + \omega_3 \frac{\partial}{\partial \theta_2}\right)^2 z &= -\frac{\partial \Phi}{\partial z},
\end{aligned} \tag{16}$$

where, from (13), the θ_i are given by

$$\theta_1 = \omega_1 t, \quad \theta_2 = \omega_2 t, \quad \theta_3 = \omega_3 t. \tag{17}$$

If we specify ω_1, ω_2 and ω_3, and treat the $\frac{\partial\Phi}{\partial x}$, $\frac{\partial\Phi}{\partial y}$ and $\frac{\partial\Phi}{\partial z}$ as functions of θ_i, the equations (16) can be viewed as non-linear equations for $x(\vec{\theta})$, $y(\vec{\theta})$ and $z(\vec{\theta})$. Now, this becomes a numerical problem which can be solved in a number of iterative ways. Ratcliff et al. chose to write the coordinates as Fourier series in the angle variables

$$\vec{x}(\vec{\theta}) = \sum_{\vec{n}} \vec{X}_n e^{i\vec{n}\cdot\vec{\theta}}. \tag{18}$$

Upon substituting (18) into (17) we obtain

$$\sum_{\vec{n}} (\vec{n}\cdot\vec{\omega})^2 \vec{X}_n e^{i\vec{n}\cdot\vec{\theta}} = \bigtriangledown\Phi, \tag{19}$$

where $\bigtriangledown\Phi$ is treated as a function of the angles. The Fourier series is then truncated after a finite number of terms and required to satisfy (19) on a grid of points around the torus. The convergence of the iteration was achieved for initial guesses sufficiently close to the exact solution.

Another approach is to divide the Hamiltonian into the separable and non-separable parts H_0 and H_1, as in equation (14). Then, one seeks a generating function that transforms the known tori of the H_0 into tori of H.

2. *Trajectory-following* methods use the Fourier analysis of the orbital trajectories to extract fundamental frequencies of the angle variables.

The method of choice for our proposed research belongs to this family of approaches, so we will say a few more words about it.

4.1. Trajectory-following Approaches

Trajectory-following algorithms take advantage of the fact that integrable motion is quasi-periodic, and can be expressed as

$$\vec{p}(t) = \sum_{\vec{n}} \vec{p}_n(\vec{J}) e^{i\vec{n}\cdot\vec{\omega}t},$$

$$\vec{q}(t) = \sum_{\vec{n}} \vec{q}_n(\vec{J}) e^{i\vec{n}\cdot\vec{\omega}t}, \tag{20}$$

where $\vec{\omega}$ is an N-D vector containing fundamental frequencies on the torus. The Fourier transform of any coordinate and its conjugated momentum will yield a frequency spectrum with spikes at discrete frequencies ω_k that are linear combinations of the fundamental frequencies. This enables us to relate the Cartesian coordinates and the angles. Let x-coordinate be given by

$$x(t) = \sum_k X_k e^{i\omega_k t} \tag{21}$$

In a triaxial potential, a 3 dof system, we have 3 fundamental frequencies. Other frequencies will be some integer combinations of these. After substituting (17) into (21), we get

$$x(t) = \sum_{lmn} X_{lmn}(J_1, J_2, J_3) e^{i(l\theta_1 + m\theta_2 + n\theta_3)} = x(\vec{J}, \vec{\theta}). \tag{22}$$

The remaining two coordinates, y and z, are given by similar expressions. The actions can be derived from Percival's formulae (Percival 1974):

$$J_1 = \sum_{lmn} l(l\omega_1 + m\omega_2 + n\omega_3)(X_{lmn}^2 + Y_{lmn}^2 + Z_{lmn}^2),$$

$$J_2 = \sum_{lmn} m(l\omega_1 + m\omega_2 + n\omega_3)(X_{lmn}^2 + Y_{lmn}^2 + Z_{lmn}^2),$$

$$J_3 = \sum_{lmn} n(l\omega_1 + m\omega_2 + n\omega_3)(X_{lmn}^2 + Y_{lmn}^2 + Z_{lmn}^2), \tag{23}$$

thus completing the transformation $(\vec{x}, \vec{v}) \mapsto (\vec{J}, \vec{\theta})$.

The similarity between the approach of Ratcliff et al. and the present one seems apparent. The decisive advantage of the present one is that its convergence does not require an initial guess close to the exact solution (in fact, since this is not an iterative scheme, it does not require one at all). Also, the implementation of the present method is computationally less involved and is guaranteed to converge to the right solution.

Now our problem of tori construction reduces to accurate calculation of fundamental frequencies and the corresponding amplitudes. The most accurate and powerful method in extracting fundamental frequencies from the orbital trajectories given in the form of a time series is Laskar's numerical analysis of fundamental frequencies (Laskar et al. 1992; Laskar 1993) (NAFF). This is one method we are going to use in our future research, so we will describe it in detail in §5.

4.2. Action Space

Having accurately determined actions for each of the orbits integrated, we can now map out the *action space*, an imaginary space whose coordinates are the actions, computed from (23). In the action space, each regular orbit is represented by a single point.

Let a small region R in action space represent a group of orbits. The volume in the 6-D phase space that these orbits occupy is

$$V = \int_D d^3\vec{x}d^3\vec{v}, \tag{24}$$

where D is the portion of the phase space visited by stars on the orbits of R. The action-angle coordinates $(\vec{J},\vec{\theta})$ are canonical, $d^3\vec{x}d^3\vec{v} = d^3\vec{J}d^3\vec{\theta}$, so

$$V = \int_D d^3\vec{J}d^3\vec{\theta}, \tag{25}$$

For any orbit, the angle variable spans the whole range $[0, 2\pi)$, so we can integrate over them:

$$V = (2\pi)^3 \int_R d^3\vec{J} = (2\pi)^3 V_R, \tag{26}$$

where V_R is the volume of the region R in the action space. Therefore, the volume of the region in the action space is directly proportional to the volume of phase space occupied by its orbits. This is a very important realization, and we will employ it to find the distribution of orbits among tubes, boxes, resonant orbits and stochastic ones.

The potentials we deal with in our research are non-integrable, so they do not admit a full set of global integrals of motion. However, a good portion of the dynamics is regular, so the above treatment can be applied. For stochastic orbits, which do not lie on tori, we can use the NAFF method described in §5 to determine the regions of the phase space that they occupy.

5. NUMERICAL ANALYSIS OF FUNDAMENTAL FREQUENCIES (NAFF)

The NAFF method extracts the fundamental frequencies of orbits with high accuracy, given the orbit in the form of the time series. For example, the fundamental frequencies in the coordinate x will be extracted using NAFF if (t_n, x_n) is specified, where t_n's are equally spaced.

The fundamental frequencies of orbits are essential for determining the occurrences of resonances, whereby one family of orbits bifurcates into another family. They are also used as a measure of orbits stochasticity, much more efficiently than the traditional method of computing the Lyapunov exponents, which necessitate long time integrations (Laskar et al. 1992).

Let

$$f(t) = \sum_{k=1}^{\infty} a_k e^{i\nu_k t} \tag{27}$$

be the quasi-periodic function of time on the interval $[-T, T]$, with the amplitudes of a_k monotonically decreasing, representing one of the orbit's coordinates, and let

$$f'(t) = \sum_{k=1}^{M} a'_k e^{i{\nu_k}'t} \tag{28}$$

be the approximation to $f(t)$ specified at N discrete equally spaced points t_n.

Look at a Fourier transform of one of the components of the series

$$g(t) = ae^{i\nu t}. \tag{29}$$

Fourier coefficients and frequencies are given by the dot product

$$\phi(\omega) = \langle g(t), e^{i\omega t}\rangle = \frac{1}{2T}\int_{-T}^{T} g(t)e^{-i\omega t}dt = \frac{a\sin(\nu-\omega)T}{(\nu-\omega)T}. \tag{30}$$

The maximum amplitude of coefficients of the spectrum will be for $\nu = \omega$.

Now, in the vicinity of the maximum of the discrete set of coefficients of the Fourier series, we can search for the true maximum of the spectrum by, for example, quadratic interpolation or bisection. This in itself improves the determination of the fundamental frequencies from the mere Fourier transform. Another way this approximation of fundamental frequencies can be improved is by "filtering" the data, in order to better isolate the peaks of fundamental frequencies. If we redefine the dot product from before as:

$$\langle f(t), g(t)\rangle = \frac{1}{2T}\int_{-T}^{T} f(t)\bar{g}(t)\chi(t)dt, \tag{31}$$

with $\chi(t) = 1 + cos(\pi t/T)$ (Hanning filter), the resulting transform of $g(t)$ will have the form

$$\phi(\omega) = \frac{-\sin{(\nu-\omega)T}}{(\nu-\omega)T[(\nu-\omega)^2T^2/\pi^2 - 1]}, \tag{32}$$

and will have more clearly defined peak of the fundamental frequency.

Clearly, the amplitude of the side lobes decrease much faster as we move away from the peaks, and the peak itself is twice as wide. This "trick" contributes to more accurate determination of the location of peaks of fundamental frequencies.

5.1. Estimating Accuracy of the NAFF

If we now analyze (28) using the NAFF method, we will obtain a second set of coefficients and frequencies:

$$f''(t) = \sum_{k=1}^{M} a_k'' e^{i\nu_k{}''t}. \tag{33}$$

Our goal is to be able to determine these fundamental frequencies as accurately and as efficiently as possible. It is desirable to determine the bound on the error in computing the fundamental frequencies in terms of the number of points N used in the FFT and the length of the integration interval T.

Quantities $\delta(a_k) = |a_k'' - a_k'|$ and $\delta(\nu_k) = |\nu_k'' - \nu_k'|$ give us a good estimate of the method's accuracy in determining amplitudes a_k' and frequencies ν_k'. If $f'(t)$ is a good approximation of $f(t)$, then these quantities can give us a good estimate of the precision of determination of amplitudes a_k and frequencies ν_k of the original function $f(t)$. Therefore, we can use the quantities $\delta(a_k)$ and $\delta(\nu_k)$ as the measure of the accuracy of the NAFF method.

5.2. Distinguishing Stochastic Orbits From Regular Using the NAFF

The NAFF method provides a reliable and efficient measure of the stochasticity of orbits, as the fundamental frequencies of regular orbits, unlike the irregular (stochastic) ones, will be constant throughout the integration. The fundamental frequencies of stochastic orbits change from one integration interval to another. This change in frequencies is used to measure Arnold diffusion, the rate at which chaotic orbits diffuse through the porous cantori in the phase space.

If the fundamental frequencies of orbits are measured given the set of close initial conditions, we can ascertain how large are the chaotic regions associated with the resonances between motion in two of the spatial coordinates. The fundamental frequencies of regular orbits started from a continuous line of initial conditions in the phase space form a continuous curve; fundamental frequencies of chaotic orbits do not. The width of the discontinuity of the plot of fundamental frequencies will be the measure of the size of the chaotic region.

6. DESCRIPTION OF THE PREVIOUS WORK

In this section we briefly summarize the previous research done by the present author in preparation for the proposed work on the doctoral dissertation.

6.1. Bifurcation of Orbits in Axisymmetric Scale-free Potentials

Our earlier study of bifurcation of orbits in axisymmetric (2 dof, with (R, z)) scale-free potentials with spheroidal equidensity surfaces. The most prominent, and thus the ones of most interest to us were bifurcations related to 1:1 and 4:3 resonances between the radial motions and motions perpendicular to the central plane. We studied these bifurcations via a class of analytic maps.

We have shown that tubelets bifurcate from the thin tube (period-1) orbit, or from the zero velocity curve, when the ratio of frequencies in R and z is a rational number (Hunter et al. 1998; Terzić 1998). In addition to these, our study suggests, the stable period-3 orbits can also arise in stable/unstable pairs via a turning point bifurcations (Terzić 1998; Hunter et al. 1998) away from the thin tube and the zero velocity curve. We were able to relate this to the transition of the rotation number of the phase-space torus through $\frac{1}{3}$ (Terzić 1998).

These minor, resonant families of orbits occupy a non-negligible portion of the phase space, and as such have to be incorporated into our self-consistent models. We expect the resonant orbits and bifurcations to play an important role in triaxial models as well.

6.2. Torus Construction Using Lie Transform Perturbation Method

One of the topics that has been of interest to us is the torus construction using the Lie perturbation method. We perturbed fully integrable isochrone potential $\Phi = \frac{-1}{1+\sqrt{1+r^2}}$, with a spherical harmonic, thus yielding a flattened axisymmetric model (Gerhard & Saha 1991). The flatness of the model depends on the expansion parameter ϵ. The perturbing potential deforms the tori of the unperturbed model in the phase space. We used Lie transforms to find a canonical transformation dependent on the perturbation parameter ϵ, which transforms the coordinates from the action-angle space of the unperturbed problem to the action-angle space of the perturbed one, in which the equations of motion simplify to (12). This transformation is given through Hamilton's equations (11), where the transforming generating function plays the role of a Hamiltonian, and the expansion parameter ϵ that of time.

The major shortcomings of the Lie transform perturbation method are its inability to deal with resonant families of orbits and the large amounts of algebraic manipulation needed to expand the perturbation potential in action-angles and to devise higher order generating function.

7. DESCRIPTION OF THE PROPOSED RESEARCH

The goal of our proposed research is to generalize our earlier work to fully triaxial models, and determine the effects of the model's departure from axial symmetry on its self-consistency and the stability of orbits. Our earlier work will, therefore, be a special case of our present study.

The previous investigations of triaxial potentials (Schwarzschild 1979; Valluri & Merritt 1998; Merritt & Valluri 1999) indicate that there are tube and box orbits present in these 3-dof systems, as well as the resonant orbits (mainly boxlets). Boxlets come in a wide variety of shapes, but in flattened triaxial models, their orbital mass densities do not contribute enough to the total density along the long axis and near the center (Zhao et al. 1999). This is in contrast to the box orbits which remain close to the axis and the center. These box orbits are necessary to build a self-consistent triaxial model, since the only remaining types of orbits, tubes, are elongated opposite to the figure of the model, and thus cannot reproduce the density distribution by themselves. This introduces a good deal of scepticism regarding the very existence of flattened triaxial scale-free models. It is one of our most important objectives to determine the constraints on triaxiality of these models imposed by the shape and the stability of their orbits. We will attempt to construct self-consistent triaxial models with variable cuspiness and flatness via Schwarzschild's method. This will give us a curve in the axis ratio space $(\frac{b}{a}, \frac{c}{a})$ separating the self-consistent models from non-self-consistent ones; ideally, this should tell us what galactic shapes we expect to see in nature. Appropriate comparison to the observational data will be presented.

7.1. Building the Self-consistent Models

The primary goal of our research is to build self-consistent triaxial scale-free models, and, as rigorously as possible, derive constraints on their triaxiality imposed by the shape and the stability of their orbits. This is why it is essential that we conduct a thorough survey of orbits in these models, as we vary their cuspiness and flatness. We outline the process of building self-consistent models:

1. Choose a potential. From (7) and (9), we see that we can vary:

    a. β, the cuspiness of the potential,

    b. lengths of axes a, b, c, which determine the flatness and the triaxiality of the model. We will systematically survey the axis ratio space $(\frac{b}{a}, \frac{c}{a})$.

2. Choose *start spaces*. These are the spaces of initial conditions which have to be chosen in such a way to feature all types of orbits presented in the potential. If the start spaces equally and thoroughly represent all of the orbits in a given potential, we expect the phase space to be a faithful reflection of the actual distribution of orbits among tubes, boxes, resonant orbits, and the stochastic ones. This is where we employ the NAFF algorithm to map out the action space and to measure the stochastic regions, in order to tell us what this distribution among orbital families is. A useful starting point may be integrating the orbits for two consecutive (equal) time intervals, and comparing the change in the action space; only the stochastic orbits are expected to change their actions over time. When the boundaries are clearly outlined, and the portions of the phase space occupied by each of the orbital families computed, we will compare this distribution to the one obtained by solving (4)-(5). This implies that we may be able to construct a self-consistent model without actually solving (4)-(5). If successful, this approach would revolutionize the construction of self-consistent models. A clever sampling of phase spaces would, thus, save us from having to do tedious and CPU-intensive computations usually associated with solving (4)-(5).

3. Integrate orbits. We will need to improve the orbit integration programs we previously used in which error of 10^{-10} was acceptable to a higher order of accuracy, about 10^{-15}. This improvement is necessary to determine the fundamental frequencies with high accuracy, so that the small fluctuations in fundamental frequencies of (weakly) stochastic orbits may be clearly distinguished from the numerical integration errors.

4. Apply the NAFF method. This will give us a set of 3 fundamental frequencies for each orbit. When we combine these into frequency maps, the regions of regular and stochastic behavior will be clearly delimited. Along the way, we will also gain some insight as to how fast do the stochastic orbits diffuse through the phase space. That, in turn, will tell us about the mixing rates of regular and stochastic orbits.

5. Build self-consistent models using Schwarzschild's method. We will integrate a large number of orbits, find their orbital densities, and solve (4)-(5). The solution will give us the number of orbits in each of the orbital families, and through some elementary algebra, the portion of the phase space that the solution predicts they occupy. We will compare this to the results of surveying the action space. Since Schwarzschild's method gives a

non-unique solution, we cannot expect the two solutions to necessarily be close. However, what we can do is "populate" orbits in Schwarzschild's method in a way consistent with the analysis of the action space and check if we obtain a self-consistent solution.

Schwarzschild (1993) built several flattened, strongly triaxial self-consistent models which included irregular orbits integrated over 1 Hubble time. Upon integrating orbits from his self-consistent models over 3 Hubble times instead, the change in orbital densities of the stochastic orbits caused the models to become more rounded. This seems to imply that strong triaxiality and flatness may be a transient feature of the observed galaxies. Our study will address this problem by finding multiple self-consistent solutions for the same model (as they are highly non-unique), integrate them for several Hubble times and check if all of them evolve toward the same shape. If so, this may be a strong indication that these models may indeed be evolving toward more axisymmetric shapes.

Since there exists a mounting evidence for the existence of central black holes in many elliptical galaxies (Ford et al. 1994; Miyoshi et al. 1995), including our own, we will also survey orbits in potentials with central point masses. These potentials will not be scale-free anymore, so additional work needs to be done.

We will also investigate the effects of rotation on the system of the morphology and regularity of orbits, and compare the results with the analysis of Merritt and Valluri (1999) who tackled this problem for a different, non-scale-free, power-law potential.

This analysis will necessarily open up more questions and issues which we will address in the course of our research.

7.2. Restricting Triaxiality Using Perturbation Methods

Along with this numerical survey, we will also employ perturbation methods to attempt to semi-analytically determine the restrictions on the triaxiality of elliptical galaxies. We will perturb a spherical potential with a triaxial one, thus creating the triaxially flattened model. The flattening will be adjusted through the perturbation parameter ϵ (see (14)). We will use Lie transform perturbation method to compute orbital densities of non-resonant orbits; for the few major resonant orbit families, we will modify this method, so that the secular effects of the resonant denominators are removed (Lichtenberg & Lieberman 1983). Then, we will check if any of these orbital densities can reproduce the total density of the model. If so, this will be a (weak) proof that the model is self-consistent. The flatness and cuspiness of models surveyed by this method will be the same as of the ones used in Schwarzschild's method, so that the two are compared on equal footing.

We expect and hope the findings of the two approaches, the Schwarzschild's method and the semi-analytical Lie transform perturbation method, to be in good agreement.

7.3. Incorporating Kinematic Data

In order to make a bridge between our theoretical work and the observations, we will compare the kinematic properties predicted by our models to the ones observed in the appropriate galaxies (the ones having the same density power-law dependence near the center and in the outer parts).

The DF of a galaxy must be reproduced from its observed properties. The self-consistent solutions for the DF produced by Schwarzschild's method are not unique, and as such will allow for a "window" of possible kinematic properties. The only observable properties are those integrated along the line of sight, at any position in the sky. In order to match the internal kinematics of a galaxy, our model must recover the galaxy's line-of-sight velocity profile (VP), defined as

$$VP(x', y', v_{z'}) = \frac{1}{\Sigma} \int \int \int f(\vec{x}, \vec{x}, \vec{v}) dz' dv_{x'} dv_{y'}, \tag{34}$$

where primed coordinates denote the system in which the line of sight is along the z'-axis. Σ is the observed surface brightness, defined as the integral of the luminous mass density $\rho_l(\vec{x}')$ along the line of sight.

In the past, the shape of the observed VP was assumed to be Gaussian, characterized by the mean velocity and the velocity dispersion (the first two moments of the VP). However, the recent advancement in the observational instruments and techniques has allowed us to go beyond the first two moments of the VP, thus realizing its departure from the Gaussian shape (Gerhard 1994). Gerhard (1993a; 1993b), van der Marel (1993) and van der Marel & Franx (1993) have independently demonstrated that the VP might best be approximated by a linear combination of Gauss-Hermite moments, best known for their use in describing the eigenfunctions of the quantum-mechanical harmonic oscillator.

The self-consistent solutions of Schwarzschild's method will give us the DF. With some algebraic manipulation of Gauss-Hermite moments, and the initial choice of the mean rotational velocity and velocity dispersion, we will be able to calculate the moments which can then be compared to the observational data.

We expect that fitting the kinematic data in addition to reproducing the flatness and triaxiality of the model will further restrict the self-consistent models in the ratio space $(\frac{b}{a}, \frac{c}{a})$.

APPENDIX

A.1. Outline of the NAFF Algorithm

The outline of the NAFF method:

1. Perform the Fast Fourier Transform (FFT) of the data (t_n, f_n). Find the maximum amplitude of the Fourier coefficient and its corresponding frequency. FFT will determine the leading (fundamental) frequency to within $d\nu = \frac{2\pi}{T}$.

2. In the vicinity of the determined maximum, find the maximum of the function relating the amplitude of the coefficients and the frequencies

$$\phi(\omega) = \langle f(t), e^{i\omega t} \rangle = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\omega t} \chi(t) dt. \tag{35}$$

Let the obtained maximum frequency be ν_j.

In the original paper, the quadratic search was used. Other searches can be used as well. A possibly computationally "cheaper" and just as effective alternative is a simple bisection search.

3. Orthonormalize the eigenvector obtained from the quadratic fit using the Gramm-Schmidt method:

$$e'_j = e^{i\nu_j t},$$

$$e_j = e'_j - \sum_{k=1}^{j-1} \langle e'_j, \tilde{e}_k \rangle \tilde{e}_k,$$

$$\tilde{e}_j = \frac{e_j}{\langle e_j, e_j \rangle}, \tag{36}$$

where $\tilde{e}_j$ is a normalized eigenvector. This yields the orthonormal set of eigenvectors onto which the function f is projected.

4. Subtract off the component of f along the newly obtained eigenvector:

$$f = f - \langle f, \tilde{e}_j \rangle \tilde{e}_j. \tag{37}$$

5. Check the exit requirement. If one of the following is satisfied:

a. $j = M$,

b. residue after j steps

$$r_j \equiv \Big| \sum_{k=1}^{\infty} a_k e^{i\nu_k t} - \sum_{k=1}^{j} a'_k e^{i\nu'_k t} \Big| \tag{38}$$

is less then some prescribed error tolerance *tol*

$$|r_j - r_{j-1}| < tol, \tag{39}$$

where *tol* is some specified error tolerance, terminate the process. Otherwise, repeat steps 1-5.

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