22 Lecture 22: Degeneracy of Matter

"Physics is very muddled again at the moment; it is much too hard for me anyway, and I wish I were a movie comedian or something like that and had never heard anything about physics!"
Wolfgang Pauli

The Big Picture: Last time we derived the Schwarzschild metric corresponding to an isolated mass, which led to the the introduction of black holes and even horizons. Today we introduce degenerate matter, such as the matter in white dwarfs and neutron stars. We also introduce polytropes as simple equilibrium stellar models.

Degeneracy

According to Pauli's Exclusion Principle, no two fermions (particles with spin of one half) can occupy the same quantum state. This is equivalent to requiring that the volume per fermion be proportional to $\lambda_c^3 \sim (\hbar/mc)^3$, where m is the fermion's mass and λ_c is its Compton wavelength. The average number density of the fermions is therefore $n_f \sim \lambda_c^{-3}$. In *white dwarfs* the density is n_f times the mass per electron, and in *neutron stars* it is the nucleon mass times n_f.

We can use this argument to compute the relative densities of white dwarfs, which are supported by electron degeneracy, and neutron stars, supported by neutron degeneracy to obtain (with approximation that the mass per electron is on the order of magnitude of the mass of the nucleon):

$$\frac{\rho_{ns}}{\rho_{wd}} = \frac{\lambda_{n,e}^{-3}}{\lambda_{c,e}^{-3}} = \frac{m_n^3}{m_e^3} = \left(\frac{m_n}{m_e}\right)^3 \approx (2000)^3 = 8 \times 10^9. \quad (426)$$

In a gas of very high fermion density, the lower momentum states are filled, so fermions must then occupy states of higher momentum. These high-momentum fermions make a large contribution to the pressure, and the gas is said to be (partially) "degenerate".

Complete Degeneracy

If the fermion density is large enough, then essentially *all* available states having energies $E < \epsilon_f$ (where ϵ_f is the *Fermi energy*, defined as the energy of the highest occupied quantum state in a system of fermions at absolute zero temperature). As the gas temperature is lowered, the distribution function

$$f(p) = \frac{1}{e^{[E(p)-\mu]/T} + 1}, \quad (427)$$

approaches unity for particle energies $E \lesssim \mu$, and zero for $E \gtrsim \mu$, where μ is the chemical potential. For $T = 0$, $\mu \equiv \epsilon_f$, so the distribution function becomes a step function:

$$f(p) = \theta(\epsilon_f - E(p)) = \begin{cases} 1 & \text{if } \epsilon_f \geq E(p), \\ 0 & \text{if } \epsilon_f < E(p). \end{cases} \quad (428)$$

The number density of fermions corresponding to the distribution function above is

$$n = g\int_0^\infty f(p)\frac{d^3p}{(2\pi\hbar)^3} = 2\int_0^{p_f} \frac{4\pi p^2 dp}{(2\pi\hbar)^3} = \frac{8\pi}{h^3}\int_0^{p_f} p^2 dp = \frac{8\pi}{h^3}\frac{1}{3}p_f^3 = \frac{8\pi}{3h^3}p_f^3, \quad (429)$$

so

$$p_f = \left(\frac{3h^3}{8\pi}n\right)^{1/3}. \quad (430)$$

p_f is the *Fermi momentum* corresponding to the Fermi energy:

$$\epsilon_f = \frac{p_f^2}{2m} \qquad \Longrightarrow \qquad p_f = \sqrt{2m\epsilon_f}. \tag{431}$$

The energy density is given by

$$\rho_e = g\int_0^\infty E(p)f(p)\frac{d^3p}{h^3} = \frac{8\pi}{h^3}\int_0^{p_f} E(p)p^2dp, \tag{432}$$

where $E(p)$ is the kinetic energy per fermion.

Nonrelativistic (complete) degeneracy. When the fermions are nonrelativistic, so $\mathbf{p} = m\mathbf{v}$ and $E(p) = p^2/2m$. The energy density then is

$$\begin{aligned} \rho &= \frac{8\pi}{h^3}\int_0^{p_f}\frac{p^2}{2m}p^2dp = \frac{8\pi}{2mh^3}\int_0^{p_f}p^4dp = \frac{8\pi}{2mh^3}\frac{1}{5}p_f^5 = \frac{8\pi}{5h^3}p_f^3\frac{p_f^2}{2m} = \frac{8\pi}{5h^3}\left(\frac{3h^3}{8\pi}n\right)\epsilon_f \\ \Longrightarrow \quad \rho &= \frac{3}{5}n\epsilon_f. \end{aligned} \tag{433}$$

For nonrelativistic particles

$$P = \frac{2}{3}\rho = \frac{2}{3}\frac{3}{5}n\epsilon_f = \frac{2}{5}n\frac{p_f^2}{2m} = \frac{n}{5m}p_f^2 = \frac{n}{5m}\left(\frac{3h^3}{8\pi}n\right)^{2/3} = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3}n^{5/3}. \tag{434}$$

The equation above is the equation of state for a nonrelativistic, completely degenerate fermion gas.

Extreme relativistic (complete) degeneracy. When the fermions are relativistic, $p \gg mc$ and $E(p) = c\sqrt{p^2 + m^2c^2} \approx cp$. The energy density then is

$$\begin{aligned} \rho &= \frac{8\pi}{h^3}\int_0^{p_f}cpp^2dp = \frac{8\pi c}{h^3}\int_0^{p_f}p^3dp = \frac{8\pi c}{h^3}\frac{1}{4}p_f^4 = \frac{2\pi}{h^3}p_f^3(cp_f) = \frac{2\pi}{h^3}\left(\frac{3h^3}{8\pi}n\right)\epsilon_f. \\ \Longrightarrow \quad \rho &= \frac{3}{4}n\epsilon_f. \end{aligned} \tag{435}$$

For relativistic particles (recall Homework set #1):

$$P = \frac{1}{3}\rho = \frac{1}{3}\frac{3}{4}n\epsilon_f = \frac{1}{4}ncp_f = \frac{1}{4}nc\left(\frac{3h^3}{8\pi}n\right)^{1/3} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}n^{4/3}. \tag{436}$$

The equation above is the equation of state for an extreme relativistic, completely degenerate fermion gas.

Important point: For complete or nearly complete degeneracy, the pressure P is independent of the temperature T.

Onset of Degeneracy

We now estimate the thresholds for the onset of the complete nonrelativistic degeneracy and complete relativistic degeneracy.

- **From nondegeneracy to complete nonrelativistic degeneracy**.

  Let us first see under which conditions will a star end up in complete nonrelativistic degeneracy. This will happen when the pressure due to the thermal equilibrium of the particles is balanced by the pressure due to the nonrelativistic degeneracy of electrons.

  Combining the equation of state for the ideal gas

$$P = \frac{\rho k T}{\bar{\mu} m_H} \tag{437}$$

  and the eq. (434), we obtain

$$\frac{\rho k T}{\bar{\mu} m_H} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \tag{438}$$

  where $\bar{\mu}$ is the mean molecular weight, defined as

$$\frac{1}{\bar{\mu}} = \sum_i \frac{\bar{n}_i m_H}{m_i}, \tag{439}$$

  m_H is the mass of the hydrogen atom, and $\bar{n}_i = \rho_i/\rho$ is the abundance of species by weight. The number density n_e of electrons is given in terms of the density as

$$n_e = \frac{\rho}{m_H \bar{\mu}_e}. \tag{440}$$

  Taking $\bar{\mu} = \bar{\mu}_e \approx 1$, the eq. (438) becomes

$$\begin{aligned}
\frac{\rho k T}{m_H} &\approx \frac{h^2}{20 m_e}\left(\frac{\rho}{m_H}\right)^{5/3} \\
\Longrightarrow \rho &= m_H \left(\frac{20 m_e k}{h^2}\right)^{3/2} T^{3/2} \\
\rho &= \left(1.67\times10^{-24}\ \mathrm{g}\right)\left(\frac{20\left(9.11\times10^{-28}\ \mathrm{g}\right)\left(1.38\times10^{-16}\ \frac{\mathrm{erg}}{\mathrm{K}}\right)}{\left(6.63\times10^{-27}\ \frac{\mathrm{erg}}{\mathrm{s}}\right)^2}\right)^{3/2} T^{3/2} \\
\rho &\approx 10^{-8} T^{3/2}.
\end{aligned} \tag{441}$$

  Therefore

$$\rho > 10^{-8} T^{3/2}, \tag{442}$$

  is the requirement for the electron gas to be completely degenerate.

- **From nonrelativistic degeneracy to extreme relativistic degeneracy**.

  In the case of relativistic particles $p_f \gg m_e c$, but the "transition" occurs at, say, $p_f = 2 m_e c$:

$$\begin{aligned}
p_f &= \left(\frac{3h^3}{8\pi} n\right)^{1/3} = \left(\frac{3h^3}{8\pi}\frac{\rho}{m_H \bar{\mu}_e}\right)^{1/3} = 2 m_e c \qquad \text{take } \bar{\mu}_e = 1 \\
\Longrightarrow \quad \rho &\approx \frac{64\pi m_H (m_e c)^3}{3h^3} \\
&= \frac{64\pi}{3}\frac{\left(1.67\times10^{-24}\ \mathrm{g}\right)\left[\left(9.11\times10^{-28}\ \mathrm{g}\right)\left(3\times10^{10}\ \frac{\mathrm{cm}}{\mathrm{s}}\right)\right]^3}{\left(6.63\times10^{-27}\ \mathrm{erg\ s}\right)^3} \\
\Longrightarrow \quad \rho &\approx 10^7\ \frac{\mathrm{g}}{\mathrm{cm}^3}.
\end{aligned} \tag{443}$$

Therefore

$$\rho > 10^7 \ \frac{\text{g}}{\text{cm}^3}. \tag{444}$$

is the requirement for the gas of electrons to reach extreme relativistic degeneracy.

These degenerate forms of matter describe brown dwarfs, white dwarfs (electron degeneracy) and neutron stars (neutron degeneracy), which we discussed in Lecture 11.

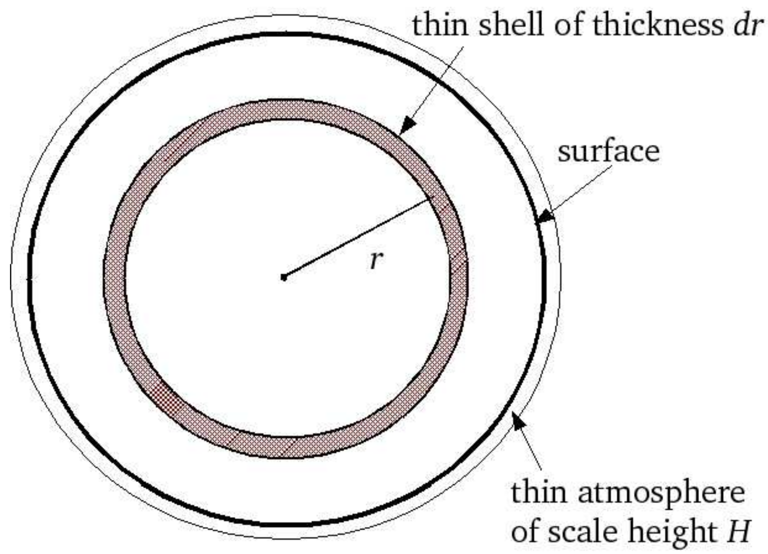

Figure 40: Simple model of a star: a sphere of gas in hydrostatic equilibrium.

Hydrostatic Equilibrium

We now present a simple model for a star in hydrostatic equilibrium.

Consider a think shell within a star in equilibrium. There are inward force acting on the shell due to its gravitating mass and the outward force of gas pressure:

$$\begin{aligned} F_g &= -G\frac{M(r)\left[\rho(r)4\pi r^2 dr\right]}{r^2} \\ F_p &= 4\pi r^2\left[P(r+dr)-P(r)\right] = 4\pi r^2 dP \end{aligned} \tag{445}$$

where $M(r)$ is mass interior to the shell:

$$M(r) = 4\pi \int_0^r \rho(\tilde{r})\tilde{r}^2 d\tilde{r}. \tag{446}$$

In hydrostatic equilibrium, these two forces are balanced, so

$$\begin{aligned} F_p &= F_g \\ 4\pi r^2 dP &= -G\frac{M(r)\left[\rho(r)4\pi r^2 dr\right]}{r^2} \\ \Longrightarrow \quad \frac{dP}{dr} &= -\rho(r)\frac{GM(r)}{r^2}. \end{aligned} \tag{447}$$

The equation above is the equation of *hydrostatic equilibrium*.

Isothermal Atmospheres in Hydrostatic Equilibrium

Stellar atmospheres are usually thin when compared to the stellar radius, which allows us to approximate the force due to gravity as a constant throughout the atmosphere:

$$g \equiv \frac{GM}{R^2} \approx const. \tag{448}$$

Let h be the height of the atmosphere (r-derivative can be replaced with an h-derivative). Then the equation of hydrostatic equilibrium [eq. (447)] then becomes

$$\frac{dP}{dh} = -\rho g. \tag{449}$$

But from the equation of state for ideal gas [eq. 437]:

$$P = \frac{\rho k T}{\bar{\mu} m_H} \qquad \Longrightarrow \qquad \rho = \frac{\bar{\mu} m_H}{kT} P, \tag{450}$$

so the eq. (449) becomes

$$\frac{dP}{dh} = -\frac{\bar{\mu} m_H g}{kT} P. \tag{451}$$

If we define the "e-folding height" ("scale height") of the atmosphere as

$$H \equiv \frac{kT}{\bar{\mu} m_H g}, \tag{452}$$

and define the initial condition $P(0) = P_0$, we can rewrite the eq. (449) and integrate it to obtain

$$\begin{aligned} \frac{dP}{dh} &= -\frac{P}{H} \qquad \Longrightarrow \qquad \frac{dP}{P} = -\frac{dh}{H} \\ \log P &= -\frac{h}{H} + c \qquad \Longrightarrow \qquad P(h) = Ce^{-h/H} \qquad \text{but} \quad P(0) = P_0 \\ \Longrightarrow \quad P(h) &= P_0 e^{-h/H}. \end{aligned} \tag{453}$$

Important point: the equation of hydrostatic equilibrium *must be* accompanied by an equation of state.

Polytropes

Polytropes are a family of equations of state for which the pressure P is given as a power of density ρ. A gas governed by a polytropic process has the equation of state

$$PV^{\gamma} = const. \tag{454}$$

Since $\rho = M/V$, where M is the mass of gas contained in volume V, we have

$$\begin{aligned} P &\propto V^{-\gamma} \propto \left(\frac{M}{\rho}\right)^{-\gamma}, \\ \Longrightarrow \quad P &= \kappa \rho^{\gamma}, \qquad \kappa = const. \end{aligned} \tag{455}$$

Gas obeying an equation of state of this form is called a *polytrope*. Examples of polytropes are given in Table 7.

Table 7: Examples of polytropic gases.

Type of polytropic gas	γ
nonrelativistic, completely degenerate gas	5/3
extreme relativistic completely degenerate gas	4/3
isothermal gas	1
gas and radiation pressure	4/3

Eddington standard model. The polytrope with $\gamma = 4/3$ is a simple model of a star supported by both radiation pressure

$$P_r = \frac{1}{3}\rho_\gamma = \frac{1}{3}\frac{\pi^2}{15}T^4 = \frac{\pi^2}{45}T^4 \equiv \frac{1}{3}aT^4, \tag{456}$$

and ideal gas pressure:

$$P_g = \frac{\rho k T}{\bar{\mu} m_H}. \tag{457}$$

Now introduce the constant β quantifying the relative contribution of gassy pressure to the total pressure (both gas and radiation) ($P = P_r + P_g$):

$$\begin{aligned} P_g &= \beta P, \qquad \Longrightarrow \qquad \beta = \frac{P_g}{P}, \\ \Longrightarrow \qquad P_r &= (1-\beta)P, \end{aligned} \tag{458}$$

so that

$$P_r = (1-\beta)P = \frac{1}{3}aT^4 \qquad \Longrightarrow \qquad T^4 = \frac{3(1-\beta)}{a}P, \tag{459}$$

Next, we eliminate the temperature T in from the equation of state:

$$\begin{aligned} \beta^4 P^4 &= P_g^4 = \left(\frac{\rho k}{\bar{\mu} m_H}\right)^4 T^4 = \left(\frac{\rho k}{\bar{\mu} m_H}\right)^4 \frac{3(1-\beta)}{a}P \\ \Longrightarrow \qquad P^3 &= \left(\frac{k}{\bar{\mu} m_H}\right)^4 \frac{3(1-\beta)}{a\beta^4}\rho^4 \\ \Longrightarrow \qquad P &= \left(\frac{k}{\bar{\mu} m_H}\right)^{4/3} \left(\frac{3(1-\beta)}{a\beta^4}\right)^{1/3} \rho^{4/3}. \end{aligned} \tag{460}$$

The term multiplying $\rho^{4/3}$ in the equation above is constant *if* β is constant (the relative breakdown of radiation and gas pressure remains unchanged) and $\bar{\mu}$ is constant (composition of gas does not change). If this is indeed the case, then we have the *Eddington standard model*

$$P = \kappa \rho^{4/3}, \qquad \kappa \equiv \left(\frac{k}{\bar{\mu} m_H}\right)^{4/3} \left(\frac{3(1-\beta)}{a\beta^4}\right)^{1/3}. \tag{461}$$

This model is a special case of Lane-Emden equations governing the polytropes in hydrostatic equilibrium which we will discuss next time.