17 Lecture 17: Recombination and Dark Matter Production

"New ideas pass through three periods:
• It can’t be done.
• It probably can be done, but it’s not worth doing.
• I knew it was a good idea all along!"

Arthur C. Clarke

The Big Picture: Today we continue discussing the recombination epoch in the early Universe. We also extend the Boltzmann formalism to the production of dark matter particles.

Recombination (continued)

Just as the neutron-nucleon ratio \( X_n \) is important to the abundance of light elements, the abundance of free electrons \( X_e \) is of great significance to the observational cosmology. Recombination, which takes place around \( z \approx 1000 \) directly leads to decoupling of photons from matter. Decoupling means that the photons stopped scattering off electrons, which become bound to neutral atoms during this epoch. The mean-free paths of photons become on the order of the size of the Universe, meaning that the Universe has become opaque. The resulting CMB radiation represents a “snapshot” of the Universe at the time of the “last scatter”.

Roughly speaking, decoupling occurs when the rate of Compton scattering of photons off electrons becomes smaller than the expansion rate of the Universe. The scattering rate is

\[
 n_e \sigma_T = X_e n_b \sigma_T, \tag{337}
\]

where \( \sigma_T = 0.665 \times 10^{-24} \text{cm}^2 \) is the Thomson cross-section, and we continue to ignore contribution of \(^4\text{He} \), by approximating \( n_e + n_H \approx n_b \). The ratio of the baryon density to the critical density is

\[
 \Omega_b \equiv \frac{\rho_b}{\rho_{cr0}} = \frac{X_e n_b}{\rho_{cr0}} = 1.87 \times 10^{-29} h^2 \text{g cm}^{-3}
\]

\[
 \Omega_b = \frac{\Omega_{b0} a^{-3}}{\rho_{cr0}} = \frac{X_e n_b}{\rho_{cr0}} = \frac{1.67 \times 10^{-24} \text{g cm}^{-3}}{h^2 \Omega_{b0} a^{-3}}
\]

\[
 \Rightarrow n_b = \rho_{cr0} \Omega_{b0} a^{-3} = \frac{1.67 \times 10^{-24} \text{g cm}^{-3}}{1.87 \times 10^{-29} h^2 \text{g cm}^{-3}} h^2 \Omega_{b0} a^{-3}
\]

so that the eq. (337) the becomes

\[
 n_e \sigma_T = 7.448 \times 10^{-30} \text{ cm}^{-1} X_e \Omega_{b0} h^2 a^{-3}. \tag{339}
\]

From eq. (73), we have

\[
 H_0 = \frac{\hbar}{0.98 \times 10^{10} \text{ years} \left( \frac{1 \text{ year}}{3600 \times 24 \times 365.25 \text{ s}} \right)} = 0.323 \times 10^{-17} \text{ s}^{-1} h,
\]

\[
 h = 3.09 \times 10^{17} s H_0, \tag{340}
\]

so that the eq. (339) can be rewritten as

\[
 n_e \sigma_T = 7.448 \times 10^{-30} \text{ cm}^{-1} X_e \Omega_{b0} h a^{-3} (3.09 \times 10^{17} s) H_0
\]

\[
 = 2.3 \times 10^{-12} \text{ s cm}^{-1} X_e \Omega_{b0} h a^{-3} H_0. \tag{341}
\]
In order to get a dimensionless equation, we multiply eq. (341) by $c/H$ (but in the equation we still omit $c$):

$$\frac{n_e \sigma_T}{H} = \left(2.3 \times 10^{-12} \text{ s cm}^{-1}\right) \left(3 \times 10^{10} \text{ cm s}^{-1}\right) X_e \Omega_{b0} h a^{-3} \frac{H_0}{H}$$

$$= 0.069 X_e \Omega_{b0} h a^{-3} \frac{H_0}{H}. \tag{342}$$

During the early epochs, the Universe is either radiation- or matter-dominated, which means that the ratio $H_0/H$ can be solved from the first Friedmann's equation [eq. (101a)]:

$$H^2 = H_0^2 \Omega_T = H_0^2 \left[\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4}\right] \implies$$

$$\frac{H}{H_0} = \left[\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4}\right]^{1/2} = \Omega_{m0}^{1/2} a^{-3/2} \left[1 + \frac{\Omega_{r0}}{\Omega_{m0}} a^{-1}\right]^{1/2} \implies$$

$$\frac{H}{H_0} = \Omega_{m0}^{1/2} a^{-3/2} \left[1 + \frac{a_{eq}}{a}\right]^{1/2}, \tag{343}$$

where we have used the results from Appendix to Lecture 9 or eqs. (2.86)-(2.87) in the textbook:

$$a_{eq} = \frac{\Omega_{r0}}{\Omega_{m0}} = \frac{4.14 \times 10^{-5}}{\Omega_{m0} h^2}. \tag{344}$$

Finally, we can rewrite eq. (342) in terms of $z$ (recall $a = 1/(1 + z)$):

$$\frac{n_e \sigma_T}{H} = 0.069 X_e \Omega_{b0} h a^{-3} \Omega_{m0}^{-1/2} a^{3/2} \left[1 + \frac{a_{eq}}{a}\right]^{-1/2}$$

$$= 0.069 X_e \Omega_{b0} h (1 + z)^{3/2} \Omega_{m0}^{-1/2} \left[1 + (1 + z) \frac{4.14 \times 10^{-5}}{\Omega_{m0} h^2}\right]^{-1/2}$$

$$= 113 X_e \left(\frac{\Omega_{b0} h^2}{0.02}\right) \left(\frac{0.15}{\Omega_{m0} h^2}\right)^{1/2} \left(\frac{1 + z}{1000}\right)^{3/2} \left[1 + \frac{1 + z}{3600} \frac{0.15}{\Omega_{m0} h^2}\right]^{-1/2}, \tag{345}$$

where the constants have been normalized to the best-fit values obtained from observations. When the free electron fraction $X_e$ drops below $\approx 10^{-2}$, photons decouple from matter. This happens before the recombination is over, i.e., before the electron fraction $X_e$ levels off below $10^{-3}$.

Even if the Universe remained ionized throughout its history, at some point photons would decouple from baryons. This can be easily seen from the eq. (345), if we set $X_e = 1$ (i.e., all electrons are free). Then, after some algebra, we arrive at

$$1 + z_{\text{decouple}} = 43 \left(\frac{0.02}{\Omega_{b0} h^2}\right)^{2/3} \left(\frac{\Omega_{m0} h^2}{0.15}\right)^{1/3}, \tag{346}$$

which, if the terms in parenthesis are taken to be equal to one, corresponds to $z_{\text{decouple}} = 42$, i.e., $t \approx 60$ million years.

**Recombination timeframe.** We can compute when the recombination took place, by computing how old the Universe was at $z \approx 1000$ (see Table 5):

$$t(z) = \frac{1}{H_0} \int_z^{\infty} \frac{d\tilde{z}}{\sqrt{\Omega_{m0}(1 + \tilde{z})^4 + \Omega_{r0}(1 + \tilde{z})^6 + \Omega_{de0}(1 + \tilde{z})^2}}. \tag{347}$$

which gives $t(1000) \approx 440,000$ years (for $h = 0.72$, $\Omega_{m0} = 0.28$, $\Omega_{r0} = 4.15 \times 10^5 h^{-2}$, $\Omega_{de0} = 0.72$).
Earlier (Appendix to Lecture 9 or eq. (2.87) in the textbook), we have derived that the Universe made a transition from radiation- to matter-dominated at about \( z_{eq} = 2.43 \times 10^4 \Omega_{m0} h^2 \approx 3500 \), which corresponds to when the Universe was about 50,000 years old. This means that the recombination happened during the matter-dominated epoch.

**Structure formation.** Recombination was followed by the *dark ages* during which the baryonic matter was neutral. It is during this time that the first structures in the Universe started to form. *Structure formation* in the Big Bang model proceeds hierarchically, with smaller structures forming before larger ones. The first structures to form are quasars, which are thought to be bright, early active galaxies, and population III stars. Before this epoch, the evolution of the Universe could be understood through linear cosmological perturbation theory — all structures could be understood as small deviations from a perfect homogeneous Universe. This is computationally relatively easy to study. At this point nonlinear structures begin to form, and the computational problem becomes much more difficult, involving, for example, N-body simulations with billions of particles.

**Reionization.** Reionization took place when the first objects started to form in the early Universe energetic enough to ionize neutral hydrogen. As these first objects formed and radiated energy, the Universe went from being neutral back to being an ionized plasma, between 150 million and one billion years after the Big Bang (at a redshift \( 6 < z < 20 \)). When protons and electrons are separate, they cannot capture energy in the form of photons. Photons may be scattered, but scattering interactions are infrequent if the density of the plasma is low. Thus, a Universe full of low density ionized hydrogen will be relatively translucent, as is the case today.
Dark Matter

Earlier, in Lectures 10 and 11, we discussed the evidence for nonbaryonic matter in the Universe, and came to the general conclusion that the total contribution of the such a matter to the energy density is \( \Omega_{dm} \approx 0.3 \). We also established WIMPs as the leading candidates for the nonbaryonic dark matter. Even though we do not know yet what these particles are, we do know that if such particles exist, they were at some point in equilibrium with the rest of the cosmic plasma at high temperatures of the early Universe. At some point, they experienced “freeze-out” as the temperature of the Universe dropped below the WIMP’s mass. Had it not been for falling out of the equilibrium (“freeze-out”), the abundance of the dark matter particles would decay as \( e^{-m/T} \), which would lead to their extinction. However, they do freeze out at some point, which is why we use the Boltzmann equation (instead of its equilibrium version, the Saha equation) to determine when they froze-out and quantify their relic abundance. The idea is to use the conclusions from observations and the earlier epochs of the Big Bang (the BBN), such as \( \Omega_{dm} \approx 0.3 \), to constrain the properties of the unknown WIMPs: their mass and cross-section. Putting such constraints on the WIMPs would be useful in the experimental attempts at their direct detection.

We now consider a generic scenario, in which two heavy WIMPs (denoted as \( X \)) annihilate and produce two light (essentially massless) particles (\( l \)). The light particles are assumed to be in complete equilibrium to the cosmic plasma, which means \( n_l = n_l(0) \). This means that in the reaction \( X + X \rightarrow l + l \) (1=1, 2=1, 3=1, 4=1), there is only one unknown \( n_X \), the abundance of the WIMPs. Again, we use the Boltzmann equation [eq. (280)]:

\[
-a^{-3} \frac{d}{dt} (n_X a^3) = n_X(0) n_X(0) \langle \sigma v \rangle \left[ \frac{n_{l1}(0) n_{l1}(0)}{n_{l1}(0) n_{l1}(0)} - \frac{n_X n_X}{n_X n_X} \right] = a^{-3} \frac{d}{dt} (n_X a^3) = \langle \sigma v \rangle \left[ \left( n_X(0) \right)^2 - \frac{1}{n_X} \right].
\]

As we did before, we continue to “massage” the Boltzmann equation into something mathematically more elucidating. After recalling that the temperature scales as \( T \sim a^{-1} \), we can rewrite the RHS of the eq. (348) above as:

\[
a^{-3} \frac{d}{dt} (n_X a^3) = a^{-3} \frac{d}{dt} \left( \frac{n_X T^3 a^3}{T^3} \right) = a^{-3} \frac{d}{dt} \left( \frac{n_X T^3}{T^3} \right) = a^{-3} \frac{d}{dt} \left( \frac{n_X T^3}{T^3} \right).
\]

After defining the quantity \( Y \) as

\[
Y \equiv \frac{n_X}{T^3},
\]

we can rewrite the eq. (348) above as

\[
T^3 \frac{dY}{dt} = \langle \sigma v \rangle T^6 \left[ \left( \frac{n_X(0)}{T^3} \right)^2 - \left( \frac{n_X}{T^3} \right)^2 \right],
\]

\[
\implies \frac{dY}{dt} = \langle \sigma v \rangle T^3 \left[ Y_{EQ}^2 - Y^2 \right],
\]

where \( Y_{EQ} \equiv n_X(0)/T^3 \). It is, again, beneficial to introduce a new time variable:

\[
x \equiv \frac{m_X}{T},
\]

where \( m_X \) is the mass of the WIMP. Again, very high temperatures correspond to \( x \ll 1 \), which is when the reactions proceed so rapidly to maintain equilibrium \( Y \approx Y_{EQ} \). Since the WIMPs
are relativistic at that time, their equilibrium abundance is given by the $m \ll T$ portion of the eq. (276), so
\[
Y \approx Y_{EQ} = \frac{n_X^{(0)}}{T^3} = \frac{g_X T^3}{T^3} = \frac{g_X}{\pi^2} \sim 1. \tag{353}
\]
For $x \gg 1$, the exponent $e^{-x}$ dominates and suppresses the equilibrium abundance $Y_{EQ}$. Eventually, the WIMPs become so rare due to this suppression that they no longer can find each other fast enough to maintain the equilibrium abundance. This is when the freeze-out begins.

We rewrite the Boltzmann equation in terms of the new integration variable $x$:
\[
\frac{dY}{dt} = \frac{dY}{dx} \frac{dx}{dt} = \frac{dY}{dx} \left[ -\frac{T}{T^3} \right] = \frac{dY}{dx} x \left[ -\frac{ka^{-2} \dot{a}}{ka-1} \right] = \frac{dY}{dx} x \left[ \dot{a} \right] = \frac{dY}{dx} x H
\]
\[
= \frac{dY}{dx} x H(x = 1) = \frac{dY}{dx} H(x = 1) = (\sigma v) T^3 \left[ Y_{EQ}^2 - Y^2 \right]
\]
\[
\Rightarrow \frac{dY}{dx} = \frac{1}{H(x = 1)} (\sigma v) T^3 \left[ Y_{EQ}^2 - Y^2 \right] = m_X^3 (\sigma v) T^3 \left[ Y_{EQ}^2 - Y^2 \right]
\]
\[
\Rightarrow \frac{dY}{dx} = -\frac{\lambda}{x^2} \left[ Y^2 - Y_{EQ}^2 \right], \tag{354}
\]
where the ratio of annihilation rate to the expansion rate is given by
\[
\lambda \equiv \frac{m_X^3 (\sigma v)}{H(x = 1)}. \tag{355}
\]
In most theories, $\lambda$ is a constant. Some theories, however, have a temperature-dependent thermally-averaged cross-section, which leads to a variable $\lambda$. This changes the quantitative results slightly, while the qualitative solutions remain the same.