5 Lecture 5: Solutions of Friedmann Equations

"A man gazing at the stars is proverbially at the mercy of the puddles in the road."

Alexander Smith

The Big Picture: Last time we derived Friedmann equations — a closed set of solutions of Einstein's equations which relate the scale factor $a(t)$, energy density ρ and the pressure P for flat, open and closed Universe (as denoted by curvature constant $k = 0, 1, -1$). Today we are going to solve Friedmann equations for the matter-dominated and radiation-dominated Universe and obtain the form of the scale factor $a(t)$. We will also estimate the age of the flat Friedmann Universe.

From the definition of the Hubble rate H in eq. (72)

$$H \equiv \frac{\dot{a}}{a} \quad \Longrightarrow \tag{102}$$

$$\dot{H} = -H^2 + \frac{\ddot{a}}{a} = -H^2\left(1 - \frac{\ddot{a}}{H^2 a}\right) \equiv -H^2(1+q), \tag{103}$$

we define a *deceleration parameter* q as

$$q \equiv -\frac{\ddot{a}}{H^2 a}. \tag{104}$$

Non-relativistic matter-dominated Universe is modeled by dust approximation: $P = 0$. Then, from eq. (95), we have

$$\frac{\ddot{a}}{a} + \frac{4\pi G}{3}\rho = 0, \tag{105}$$

and, in terms of H

$$-H^2 q + \frac{4\pi G}{3}\rho = 0. \tag{106}$$

Therefore

$$\rho = \frac{3H^2}{4\pi G} q. \tag{107}$$

Then the first Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho = -\frac{k}{a^2},$$

$$H^2 - 2H^2 q = -\frac{k}{a^2}, \tag{108}$$

so

$$-k = a^2 H^2 (1 - 2q). \tag{109}$$

Since both $a \neq 0$ and $H \neq 0$, for flat Universe ($k = 0$), $q = 1/2$ ($q > 1/2$ for $k = 1$ and $q < 1/2$ for $k = -1$). When combined with eq. (107), this yields *critical density*

$$\rho_{\rm cr} = \frac{3H^2}{8\pi G}, \tag{110}$$

the density needed to yield the flat Universe. Currently, it is (see eq. (73))

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi G} = \frac{3\left(\frac{h}{0.98\times 10^{10}\ \rm years}\right)^2\left(\frac{1\ \rm year}{3600\times 24\times 365\ \rm sec}\right)^2}{8\pi\left(6.67\times 10^{-8}{\rm cm}^3\ {\rm g}^{-1}\ {\rm s}^{-2}\right)} = 1.87\times 10^{-29}h^2\ \frac{\rm g}{\rm cm^3} \approx 10^{-29}\ \frac{\rm g}{\rm cm^3}.$$

(We used $h \approx 0.72 \pm 0.02$.)

It is important to note that the quantity q provides the relationship between the density of the Universe ρ and the critical density $\rho_{\rm cr}$ (after combining eqs. (107) and (109)):

$$q = \frac{\rho}{2\rho_{\rm cr}}. \tag{111}$$

The second Friedmann equation (eq. (101b)) for the matter-dominated Universe becomes

$$\begin{aligned} \dot\rho + 3\rho\frac{\dot a}{a} &= 0 \\ a^3\dot\rho + 3\rho\dot a a^2 &= 0 \quad \Rightarrow \quad \frac{d}{dt}\left(a^3\rho\right) = 0 \quad \Rightarrow \quad a^3\rho = a_0^3\rho_0 = const. \end{aligned} \tag{112}$$

Radiation-dominated Universe is modeled by perfect fluid approximation with $P = \frac{1}{3}\rho$.

The second Friedmann equation (eq. (101b)) becomes

$$\begin{aligned} \dot\rho + 3\left(\rho + \frac{1}{3}\rho\right)\frac{\dot a}{a} &= \dot\rho + 4\rho\frac{\dot a}{a} = 0 \\ a^4\dot\rho + 4\rho\dot a a^3 &= 0 \quad \Rightarrow \quad \frac{d}{dt}\left(a^4\rho\right) = 0 \quad \Rightarrow \quad a^4\rho = a_0^4\rho_0 = const. \end{aligned} \tag{113}$$

Flat Universe ($k = 0,\ q_0 = \frac{1}{2}$)

Matter-dominated (dust approximation): $P = 0,\ \ a^3\rho = const.$

The first Friedmann equation (eq. (101a)) becomes

$$\begin{aligned} &\frac{\dot a^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 \\ \Rightarrow\quad &\frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\frac{1}{a^{1/2}} \quad \Rightarrow \quad \int a^{1/2}da = \frac{2}{3}a^{3/2} + K = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\ t. \end{aligned} \tag{114}$$

At the Big Bang, $t = 0$, $a = 0$, so $K = 0$. Upon adopting convention $a_0 = 1$, and the fact that the Universe is flat $\rho_0 = \rho_{\rm cr}$, we finally have

$$\begin{aligned} a &= \left(6\pi G\rho_0\right)^{1/3}t^{2/3} = \left(6\pi G\rho_{\rm cr}\right)^{1/3}t^{2/3} \\ &= \left(6\pi G\frac{3H_0^2}{8\pi G}\right)^{1/3}t^{2/3} = \left(\frac{9H_0^2}{4}\right)^{1/3}t^{2/3} = \left(\frac{3H_0}{2}\right)^{2/3}t^{2/3}. \end{aligned} \tag{115}$$

where we have used the eq. (110) in the second step. From here we compute the age of the Universe t_0, which corresponds to the Hubble rate H_0 and the scale factor $a = a_0 = 1$ to be:

$$t_0 = \frac{2}{3H_0}. \tag{116}$$

Taking $H_0 = \frac{h}{0.98\times 10^{10}\ \rm years}$ and $h \approx 72$, we get

$$t_0 = \frac{2\times 0.98\times 10^{10}\ \rm years}{3\times 0.72} \approx 9.1\times 10^9\ {\rm years} \equiv 9.1\ \mathcal{A}\ \ (\rm aeon). \tag{117}$$

Radiation-dominated: $P = \frac{1}{3}\rho, \quad a^4\rho = const.$

The first Friedmann equation (eq. (101a)) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^4$$

$$\Rightarrow \quad \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3}}\frac{1}{a} \quad \Rightarrow \quad \int a da = 2a^2 + K = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3}}\, t. \tag{118}$$

Again, at the Big Bang, $t = 0$, $a = 0$, so $K = 0$, and a_0=1. Also $\rho_0 = \rho_{\rm cr}$. Therefore,

$$a = \left(\frac{2}{3}\pi G\rho_0\right)^{1/4} t^{1/2} = \left(\frac{2}{3}\pi G\rho_{\rm cr}\right)^{1/4} t^{1/2} = \left(\frac{2}{3}\pi G\frac{3H_0^2}{8\pi G}\right)^{1/4} t^{1/2} = \left(\frac{H_0}{2}\right)^{1/2} t^{1/2}. \tag{119}$$

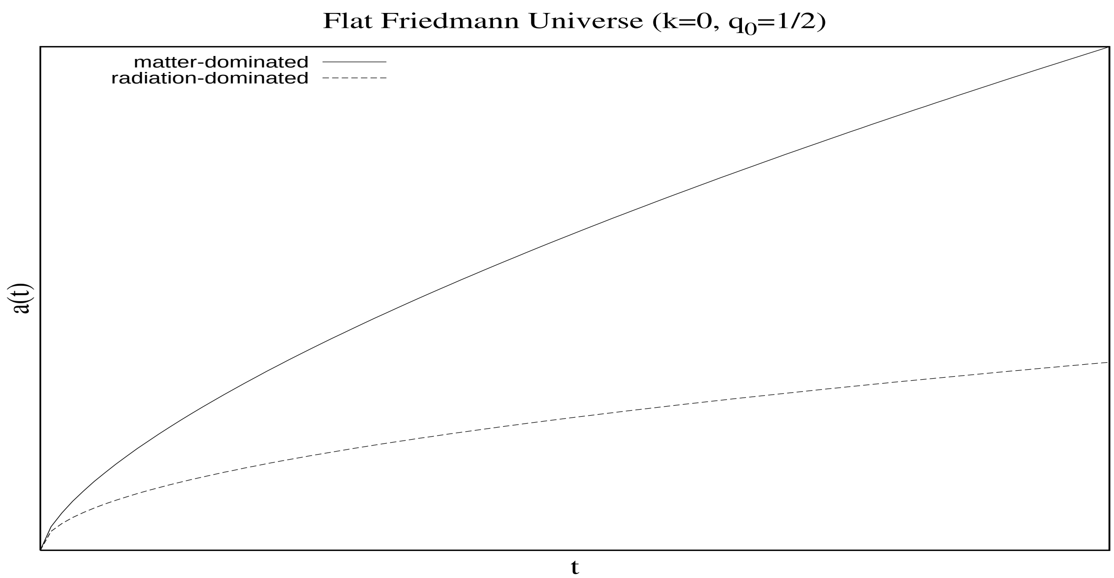

Figure 2: Evolution of the scale factor $a(t)$ for the flat Friedmann Universe.

Closed Universe ($k = 1, \ q_0 > \frac{1}{2}$)

Matter-dominated (dust approximation): $P = 0, \quad a^3\rho = const.$

The first Friedmann equation (eq. (101a)) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 - \frac{1}{a^2}$$

$$\Rightarrow \quad \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} - 1} \quad \Rightarrow \quad \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} - 1}}$$

Rewrite the integral above in terms of conformal time given in eq. (83) ($d\eta \equiv \frac{dt}{a}$):

$$\int d\eta = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a - a^2}}, \tag{120}$$

and define, after substituting $a_0 = 1$ and using eqs. (107)-(109)

$$A \equiv \frac{4\pi G\rho_0}{3} = H_0^2 q_0 = \frac{q_0}{2q_0 - 1}. \tag{121}$$

Then

$$\eta - \eta_0 = \int_0^a \frac{d\tilde{a}}{\sqrt{2A\tilde{a} - \tilde{a}^2}} = \sin^{-1}\left(\frac{a-A}{A}\right) + \frac{1}{2}\pi. \tag{122}$$

But, the requirement $\eta = 0$ at $a = 0$ sets $\eta_0 = 0$, so we have

$$\frac{a-A}{A} = \sin\left(\eta - \frac{1}{2}\pi\right) = -\cos\eta \qquad \Rightarrow \qquad a = A(1-\cos\eta). \tag{123}$$

Now $dt = a d\eta$, so

$$t - t_0 = \int a d\eta = \int A(1-\cos\eta)d\eta = A\int(1-\cos\eta)\ d\eta = A(\eta - \sin\eta). \tag{124}$$

But, the requirement $\eta = 0$ at $t = 0$ sets $t_0 = 0$. Therefore, we finally have the dependence of the scale factor a in terms of the time t parametrized by the conformal time η as:

$$\begin{aligned} a &= \frac{q_0}{2q_0-1}(1-\cos\eta), \\ t &= \frac{q_0}{2q_0-1}(\eta - \sin\eta). \end{aligned} \tag{125}$$

Radiation-dominated: $P = \frac{1}{3}\rho, \quad a^4\rho = const.$

The first Friedmann equation (eq. (101a)) becomes

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^4 - \frac{1}{a^2} \\ \Rightarrow \qquad \frac{da}{dt} &= \sqrt{\frac{8\pi G\rho_0 a_0^4}{3a^2} - 1} \qquad \Rightarrow \qquad \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a^2} - 1}} \end{aligned}$$

Again, rewrite the integral above in terms of conformal time and quantity $A_1 = \frac{8\pi G\rho_0}{3} = \frac{2q_0}{2q_0-1}$:

$$\eta - \eta_0 = \int_0^a \frac{d\tilde{a}}{\sqrt{A_1 - \tilde{a}^2}} = \sin^{-1}\left(\frac{a}{\sqrt{A_1}}\right). \tag{126}$$

Again, the requirement $\eta = 0$ at $a = 0$ sets $\eta_0 = 0$, so we have

$$a = \sqrt{A_1}\sin(\eta), \tag{127}$$

and

$$t - t_0 = \sqrt{A_1}\cos(\eta), \tag{128}$$

The requirement $\eta = 0$ at $t = 0$ sets $t_0 = \sqrt{A_1}$, so we finally have

$$\begin{aligned} a &= \sqrt{\frac{2q_0}{2q_0-1}}\sin\eta, \\ t &= \sqrt{\frac{2q_0}{2q_0-1}}(1-\cos\eta). \end{aligned} \tag{129}$$

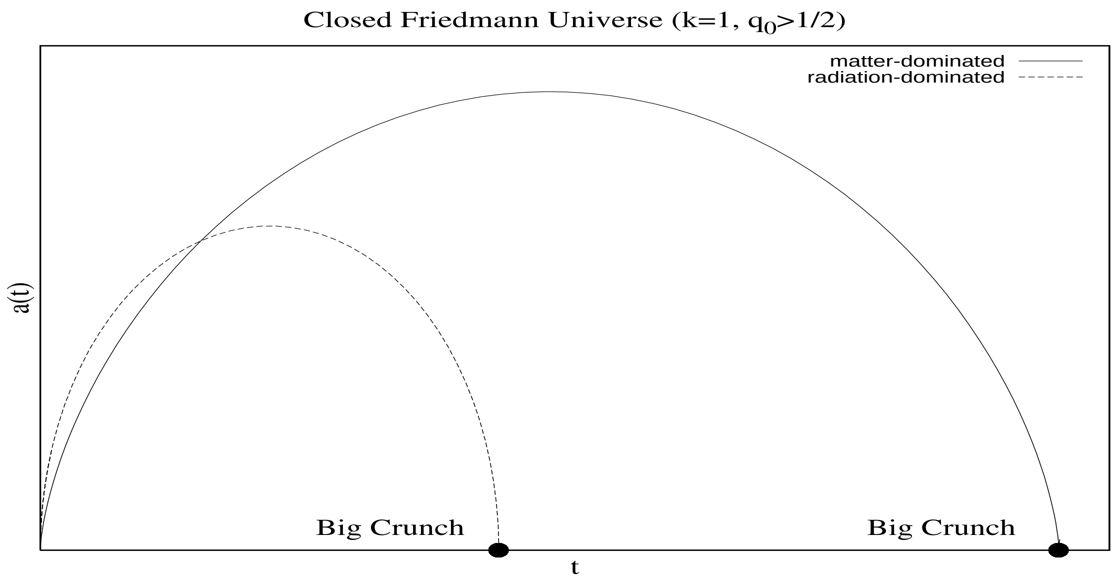

Figure 3: Evolution of the scale factor $a(t)$ for the closed Friedmann Universe.

In both matter- and radiation-dominated closed Universes, the evolution is cycloidal — the scale factor grows at an ever-decreasing rate until it reaches a point at which the expansion is halted and reversed. The Universe then starts to compress and it finally collapses in the Big Crunch.

Open Universe ($k = -1,\ q_0 < \frac{1}{2}$)

Matter-dominated (dust approximation): $P = 0,\quad a^3\rho = const.$

The first Friedmann equation (eq. (101a)) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 + \frac{1}{a^2}$$

$$\Rightarrow \qquad \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} + 1} \quad \Rightarrow \quad \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} + 1}}$$

Again, rewrite the integral above in terms of conformal time:

$$\int d\eta = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a + a^2}}, \tag{130}$$

take $a_0 = 1$, and define $\tilde{A} \equiv \frac{4\pi G\rho_0}{3} = \frac{q_0}{2q_0 - 1}$. Then

$$\begin{aligned}\eta - \eta_0 &= \int_0^a \frac{d\tilde{a}}{\sqrt{2\tilde{A}\tilde{a} + \tilde{a}^2}} = \ln\left(\frac{a + \tilde{A} + \sqrt{a(2\tilde{A} + a)}}{\tilde{A}}\right) = \ln\left(\frac{a}{\tilde{A}} + 1 + \sqrt{2\frac{a}{\tilde{A}} + \left(\frac{a}{\tilde{A}}\right)^2}\right) \\ &= \cosh^{-1}\left(\frac{a}{\tilde{A}} + 1\right).\end{aligned} \tag{131}$$

But, the requirement $\eta = 0$ at $a = 0$ sets $\eta_0 = 0$, so we have

$$\frac{a + \tilde{A}}{\tilde{A}} = \cosh\eta \qquad \Rightarrow \qquad a = \tilde{A}(\cosh\eta - 1). \tag{132}$$

Now $dt = ad\eta$, so

$$t - t_0 = \int a d\eta = \int \tilde{A}(\cosh\eta - 1)d\eta = \tilde{A}\int(\cosh\eta - 1)\ d\eta = \tilde{A}(\sinh\eta - \eta). \quad (133)$$

But, the requirement $\eta = 0$ at $t = 0$ sets $t_0 = 0$. Therefore, we finally have the dependence of the scale factor a in terms of the time t parametrized by the conformal time η as:

$$\begin{aligned} a &= \frac{q_0}{2q_0 - 1}(\cosh\eta - 1), \\ t &= \frac{q_0}{2q_0 - 1}(\sinh\eta - \eta). \end{aligned} \quad (134)$$

Radiation-dominated: $P = \frac{1}{3}\rho, \quad a^4\rho = const.$

The first Friedmann equation (eq. (101a)) becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^4 + \frac{1}{a^2}$$

$$\Rightarrow \quad \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3a^2} + 1} \quad \Rightarrow \quad \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a^2} + 1}}$$

Again, rewrite the integral above in terms of conformal time and quantity $\tilde{A}_1 \equiv \frac{8\pi G\rho_0}{3} = \frac{2q_0}{2q_0 - 1}$:

$$\eta - \eta_0 = \int_0^a \frac{d\tilde{a}}{\sqrt{\tilde{A}_1 + \tilde{a}^2}} = \sinh^{-1}\left(\frac{a}{\sqrt{\tilde{A}_1}}\right) \quad (135)$$

Again, the requirement $\eta = 0$ at $a = 0$ sets $\eta_0 = 0$, so we have

$$a = \sqrt{\tilde{A}_1}\sinh\eta, \quad (136)$$

$$t - t_0 = \sqrt{\tilde{A}_1}\cosh\eta, \quad (137)$$

The requirement $\eta = 0$ at $t = 0$ sets $t_0 = \sqrt{\tilde{A}_1}$, so we finally have

$$\begin{aligned} a &= \sqrt{\frac{2q_0}{1 - 2q_0}}\sinh\eta, \\ t &= \sqrt{\frac{2q_0}{1 - 2q_0}}\left(\cosh\eta - 1\right). \end{aligned} \quad (138)$$

Early times (small η limit): For small values of η, the trigonometric and hyperbolic functions can be expanded in Taylor series (keeping only first two terms):

$$\begin{aligned} \sin\eta &= \eta - \frac{1}{6}\eta^3, \qquad \cos\eta = 1 - \frac{1}{2}\eta^2, \\ \sinh\eta &= \eta + \frac{1}{6}\eta^3, \qquad \cosh\eta = 1 + \frac{1}{2}\eta^2, \end{aligned}$$

so, to the leading term, the a and t dependence on η for the different curvatures is shown in the table below:

Moral: at early times, the curvature of the Universe does not matter — singular behavior at early times is essentially independent of the curvature of the Universe (k). Big Bang — "matter-dominated singularity".

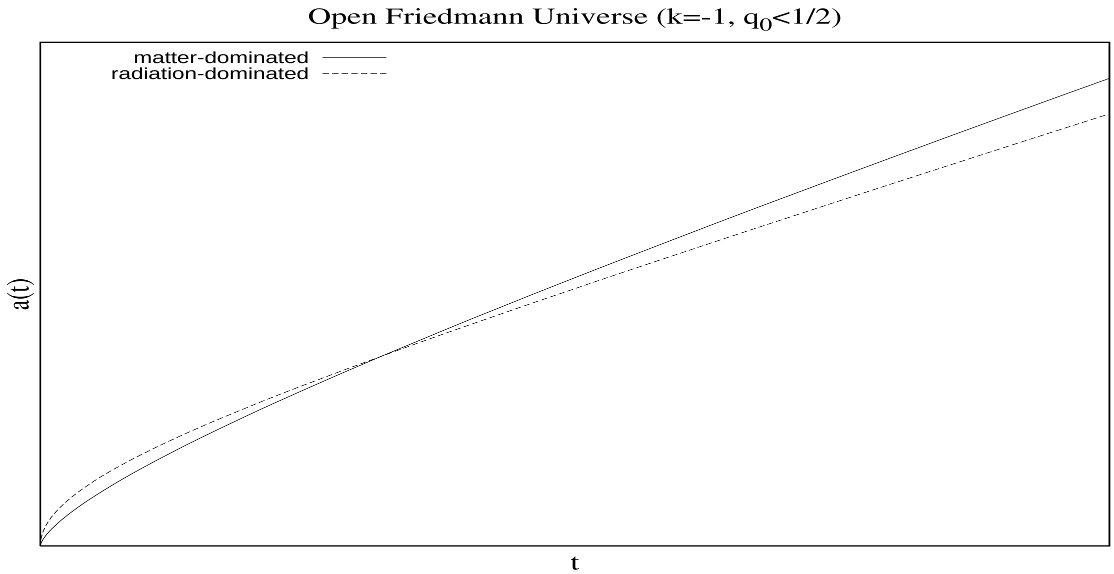

Figure 4: Evolution of the scale factor $a(t)$ for the open Friedmann Universe.

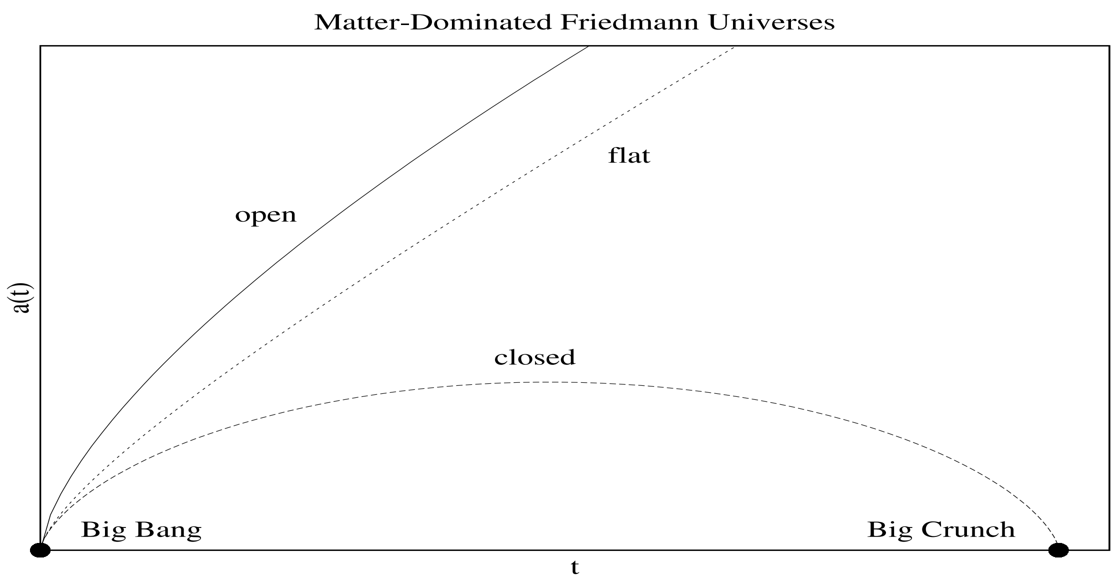

Figure 5: Evolution of the scale factor $a(t)$ for the flat, closed and open matter-dominated Friedmann Universes.

Table 2: Scale factor $a(t)$ for flat, closed and open Friedmann Universes, along with their asymptotic behavior at early times.

curvature	For all η		For small η		
k	a	t	a	t	$a(t)$
0	$(6\pi G\rho_0)^{1/3}\, t^{2/3}$	-	$\propto t^{2/3}$	-	$\propto t^{2/3}$
1	$\frac{q_0}{2q_0-1}(1-\cos\eta)$	$\frac{q_0}{2q_0-1}(\eta-\sin\eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$
-1	$\frac{q_0}{1-2q_0}(\cosh\eta-1)$	$\frac{q_0}{1-2q_0}(\sinh\eta-\eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$