Appendix to Lecture 6

Matter–Dark Energy Equality

In class, a question was raised of when was the energy density of matter equal to the “vacuum” (dark) energy density.

This can be computed easily after recalling that

\[
\rho_{\text{de}} = \text{const.} = \rho_{\text{de}0},
\rho_{\text{m}}a^3 = \text{const.}, \quad \Rightarrow \quad \rho_{\text{m}}a^3 = \rho_{\text{m}0}a_0^3 \quad \Rightarrow \quad \rho_{\text{m}} = \rho_{\text{m}0}a^{-3},
\]

after noting that, by convention, \(a_0 = 1\). So, the two energy densities are equal at \(a_{\text{eq}2}\) when

\[
1 = \frac{\rho_{\text{de}}}{\rho_{\text{m}}} = \frac{\rho_{\text{de}0}}{\rho_{\text{m}0}a_{\text{eq}2}^{-3}},
\]

\[
\Rightarrow a_{\text{eq}2} = \left(\frac{\rho_{\text{m}0}}{\rho_{\text{de}0}}\right)^{1/3} = \left(\frac{0.28}{0.72}\right)^{1/3} = 0.73.
\]

So, the energy density of matter and the energy density of dark energy were equal when the Universe was 0.73 — almost 3/4 — of its size today.

To compute how long ago this took place, we can compute the age of the Universe at \(a_{\text{eq}2}\) from eq. (156)

\[
H_0t_0 = \int_0^{a_{\text{eq}2}} \frac{da}{\sqrt{\frac{1-\Omega_{\text{de}0}}{a} + \Omega_{\text{de}0}a^2}} = \int_0^{a_{\text{eq}2}} \frac{a^{1/2}da}{\sqrt{(1 - \Omega_{\text{de}0}) + \Omega_{\text{de}0}a^3}}
\]

\[
= \frac{2}{3\sqrt{\Omega_{\text{de}0}}} \ln \left[ 2 \left( \sqrt{\Omega_{\text{de}0}a^3} + \sqrt{\Omega_{\text{de}0}(a^3 - 1) + 1} \right) \right]_0^{a_{\text{eq}2}},
\]

by changing the upper limits of integration from \(t_0\) and \(a(t_0) = 1\) to \(t_1\) and \(a(t_1) \equiv a_{\text{eq}2}\):

\[
H_0t_1 = \int_0^{a_{\text{eq}2}} \frac{da}{\sqrt{\frac{1-\Omega_{\text{de}0}}{a} + \Omega_{\text{de}0}a^2}} = \int_0^{a_{\text{eq}2}} \frac{a^{1/2}da}{\sqrt{(1 - \Omega_{\text{de}0}) + \Omega_{\text{de}0}a^3}}
\]

\[
= \frac{2}{3\sqrt{\Omega_{\text{de}0}}} \ln \left[ \sqrt{\Omega_{\text{de}0}a_{\text{eq}2}^3} + \sqrt{\Omega_{\text{de}0}(a_{\text{eq}2}^3 - 1) + 1} \right]_0^{a_{\text{eq}2}}
\]

\[
= \frac{2}{3\sqrt{\Omega_{\text{de}0}}} \ln \left[ \sqrt{\Omega_{\text{de}0}a_{\text{eq}2}^3 eq2 + \sqrt{\Omega_{\text{de}0}(a_{eq2}^3 - 1) + 1}} \right].
\]

So, for the observed parameters of \(\Omega_{\text{de}} = 0.72\) and the computed value of \(a_{\text{eq}2} = 0.73\), we obtain

\[
t_1 = \frac{2}{3H_0\sqrt{\Omega_{\text{de}0}}} \ln \left[ \sqrt{0.72 \cdot 0.73^3} + \sqrt{0.72(0.73^3 - 1) + 1} \right] = \frac{2}{3H_0\sqrt{\Omega_{\text{de}0}}} (0.881).
\]

We compare this to the age of the Universe computed earlier in eq. (157)

\[
t_0 = \frac{2}{3H_0\sqrt{\Omega_{\text{de}0}}} \ln \left( \frac{1 + \sqrt{\Omega_{\text{de}0}}}{\sqrt{1 - \Omega_{\text{de}0}}} \right) = \frac{2}{3H_0\sqrt{\Omega_{\text{de}0}}} (1.25) = 13.7A.
\]
and compare to finally obtain
\[
\frac{t_1}{0.867} = \frac{t_0}{1.25} \Rightarrow t_1 = \frac{0.867}{1.25} t_0 = 0.69 t_0 = 9.65 \, A.
\]

So, the Universe was 9.65 billion years old when energy densities of matter and dark energy were equal. That was \(13.7 - 9.65 = 4.05\) billion years ago.

Figure 17: Three epochs in the evolution of the Universe: (1) radiation-dominated \(a < a_{eq}\), (2) matter-dominated \(a_{eq} < a < a_{eq2}\), (3) dark energy-dominated \(a > a_{eq2}\). For the preview of what processes are occurring in each of these epochs, see Fig. 1.15 in the textbook.