Appendix to Lecture 6

Matter–Dark Energy Equality

In class, a question was raised of when was the energy density of matter equal to the "vacuum" (dark) energy density.

This can be computed easily after recalling that

$$\begin{aligned} \rho_{\rm de} &= const. = \rho_{\rm de0}, \\ \rho_{\rm m} a^3 &= const., \qquad \Rightarrow \qquad \rho_{\rm m} a^3 = \rho_{\rm m0} a_0^3 \qquad \Rightarrow \qquad \rho_{\rm m} = \rho_{\rm m0} a^{-3}, \end{aligned}$$

after noting that, by convention, $a_0 = 1$. So, the two energy densities are equal at $a_{\rm eq2}$ when

$$\begin{aligned} 1 &= \frac{\rho_{\rm de}}{\rho_{\rm m}} = \frac{\rho_{\rm de0}}{\rho_{\rm m0} a_{\rm eq2}^{-3}}, \\ \Rightarrow & \quad a_{\rm eq2} = \left(\frac{\rho_{\rm m0}}{\rho_{\rm de0}}\right)^{1/3} = \left(\frac{0.28}{0.72}\right)^{1/3} = 0.73. \end{aligned}$$

So, the energy density of matter and the energy density of dark energy were equal when the Universe was 0.73 — almost 3/4 — of its size today.

To compute how long ago this took place, we can compute the age of the Universe at $a_{\rm eq2}$ from eq. (156)

$$\begin{aligned} H_0 t_0 &= \int_0^1 \frac{da}{\sqrt{\frac{1-\Omega_{\rm de0}}{a} + \Omega_{\rm de0} a^2}} = \int_0^1 \frac{a^{1/2} da}{\sqrt{(1-\Omega_{\rm de0}) + \Omega_{\rm de0} a^3}} \\ &= \frac{2}{3\sqrt{\Omega_{\rm de0}}} \ln \left[2 \left(\sqrt{\Omega_{\rm de0} a^3} + \sqrt{\Omega_{\rm de0}(a^3 - 1) + 1} \right) \right] \Bigg|_0^1, \end{aligned}$$

by changing the upper limits of integration from t_0 and $a(t_0) = 1$ to t_1 and $a(t_1) \equiv a_{\rm eq2}$:

$$\begin{aligned} H_0 t_1 &= \int_0^{a_{\rm eq2}} \frac{da}{\sqrt{\frac{1-\Omega_{\rm de0}}{a} + \Omega_{\rm de0} a^2}} = \int_0^{a_{\rm eq2}} \frac{a^{1/2} da}{\sqrt{(1-\Omega_{\rm de0}) + \Omega_{\rm de0} a^3}} \\ &= \frac{2}{3\sqrt{\Omega_{\rm de0}}} \ln \left[2 \left(\sqrt{\Omega_{\rm de0} a^3} + \sqrt{\Omega_{\rm de0}(a^3 - 1) + 1} \right) \right] \Bigg|_0^{a_{\rm eq2}} \\ &= \frac{2}{3\sqrt{\Omega_{\rm de0}}} \ln \left[\frac{\sqrt{\Omega_{\rm de0} a_{\rm eq2}^3} + \sqrt{\Omega_{\rm de0}\left(a_{\rm eq2}^3 - 1\right) + 1}}{\sqrt{1-\Omega_{\rm de0}}} \right]. \end{aligned}$$

So, for the observed parameters of $\Omega_{\rm de} = 0.72$ and the computed value of $a_{\rm eq2} = 0.73$, we obtain

$$t_1 = \frac{2}{3H_0\sqrt{\Omega_{\rm de0}}} \ln \left[\frac{\sqrt{0.72\ 0.73^3} + \sqrt{0.72\left(0.73^3 - 1\right) + 1}}{\sqrt{1-0.72}} \right] = \frac{2}{3H_0\sqrt{\Omega_{\rm de0}}}(0.881).$$

We compare this to the age of the Universe computed earlier in eq. (157)

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{\rm de0}}} \ln \left(\frac{1+\sqrt{\Omega_{\rm de0}}}{\sqrt{1-\Omega_{\rm de0}}} \right) = \frac{2}{3H_0\sqrt{\Omega_{\rm de0}}}(1.25) = 13.7\mathcal{A}.$$

and compare to finally obtain

$$\frac{t_1}{0.867} = \frac{t_0}{1.25} \qquad \Rightarrow \qquad t_1 = \frac{0.867}{1.25} t_0 = 0.69 t_0 = 9.65\mathcal{A}.$$

So, the Universe was 9.65 billion years old when energy densities of matter and dark energy were equal. That was $13.7 - 9.65 = 4.05$ billion years ago.

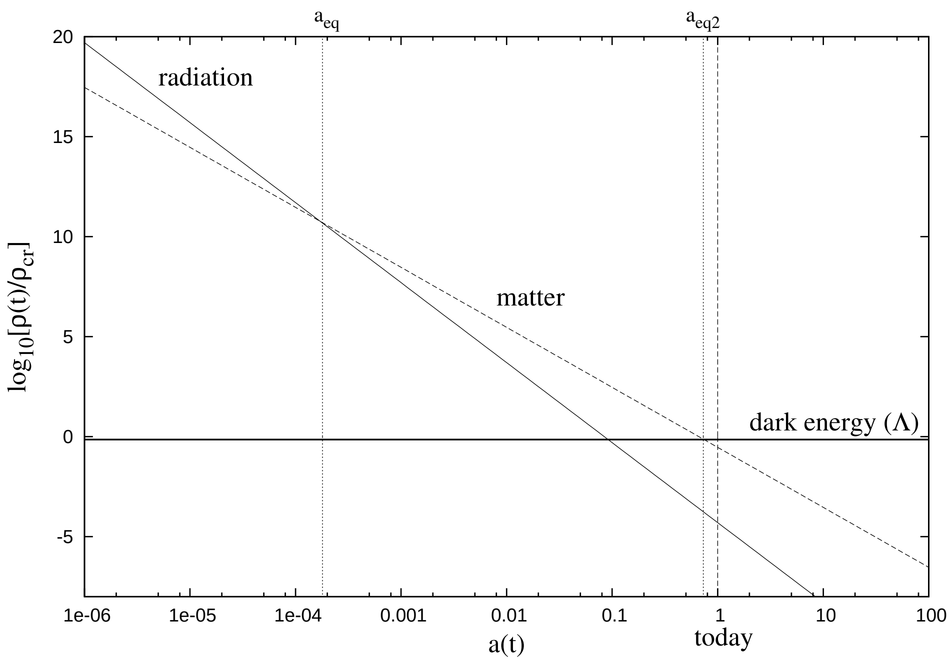

Figure 17: Three epochs in the evolution of the Universe: (1) radiation-dominated $a < a_{\rm eq}$, (2) matter-dominated $a_{\rm eq} < a < a_{\rm eq2}$, (3) dark energy-dominated $a > a_{\rm eq2}$. For the preview of what processes are occurring in each of these epochs, see Fig. 1.15 in the textbook.