Longitudinal and transverse emittance exchange: transfer matrix and phase space

Y.-E Sun
(Dated: February 15, 2007)

In this note, the transfer matrix of the emittance exchange optics composed of a double dogleg with a dipole-mode cavity in the middle point is explicitly written out. Beamline dispersion and momentum compaction are derived as an expression of the beamline setup. Several groups of representative particles are tracked through the emittance exchanger and the corresponding phase space snap shots are taken at different longitudinal locations of interests.

I. TRANSFER MATRIX

The 4-D phase space coordinates can be represented by a 4-D vector U,

$$U = \begin{bmatrix} x \\ x' \\ z \\ \delta \end{bmatrix}. \tag{1}$$

The transfer matrix of a dipole bending magnet with bending angle θ and radius ρ is given by

$$M(\theta, \rho) = \begin{bmatrix} \cos\theta & \rho\sin\theta & 0 & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & 0 & \sin\theta \\ -\sin\theta & -\rho(1-\cos\theta) & 1 & -\rho(\theta-\sin\theta) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Under thin-lens approximation, it becomes [1]

$$M_{th}(\theta, \rho) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta \\ -\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

The transfer matrix of a drift of distance L is

$$Dft(L) = \begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Under the thin kens approximation, a doglog can be represented by

$$Dogleg(\theta, \rho, L) = M_{th}(\theta, \rho) \cdot Dft(L) \cdot M_{th}(-\theta, -\rho) = \begin{bmatrix} 1 & L & 0 & -\theta L \\ 0 & 1 & 0 & 0 \\ 0 & -\theta L & 1 & \theta^2 L \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Compared with Eq. (19) in Ref. [2], we establish the following relations:

$$\eta = -\theta L, \tag{6}$$
$$\xi = \theta^2 L. \tag{7}$$

The transfer matrix of a dipole mode cavity under thin lens approximation is

$$Cavity(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

The total matrix for the emittance exchanger is

$$M_{total}(\theta,\rho,L,S,k) = Dogleg(\theta,\rho,L)\cdot Dft(S)\cdot Cavity(k)\cdot Dft(S)\cdot Dogleg(\theta,\rho,L), \tag{9}$$

where S is the distance between the middle dipole magnet and cavity. Under the emittance-exchange condition $1+k\eta=0$ [2] , we have

$$k = -\frac{1}{\eta} = \frac{1}{\theta L}. \tag{10}$$

Substitute Eq. 10 into Eq. 9, we have

$$M_{total} = \begin{bmatrix} 0 & 0 & -\frac{S+L}{\theta L} & \theta S \\ 0 & 0 & \frac{1}{\theta L} & \theta \\ \theta & \theta S & 0 & 0 \\ \frac{1}{\theta L} & \frac{1}{\theta}(1+\frac{S}{L}) & 0 & 0 \end{bmatrix}. \tag{11}$$

The total transfer matrix given by Eq. 11 is anti-block-diagonal as expected under the total emittance exchange condition.

II. PHASES SPACE SNAP SHOTS

[1] K.-J. Kim, P. Piot, J. Power, discussions.
[2] P. Emma, Z. Huang, K.-J. Kim and P. Piot, Phy. Rev. ST Accel. Beams **9**, 100702, 2006.
[3] M. Cornacchia and P. Emma, Phy. Rev. ST Accel. Beams **5**, 084001, 2002. E_z and B_y given by Eq. (1) and (2) are $E_z = E_0\frac{x}{a}\cos(\omega t), B_y = \frac{E_0}{a\omega}\sin(\omega t)$.

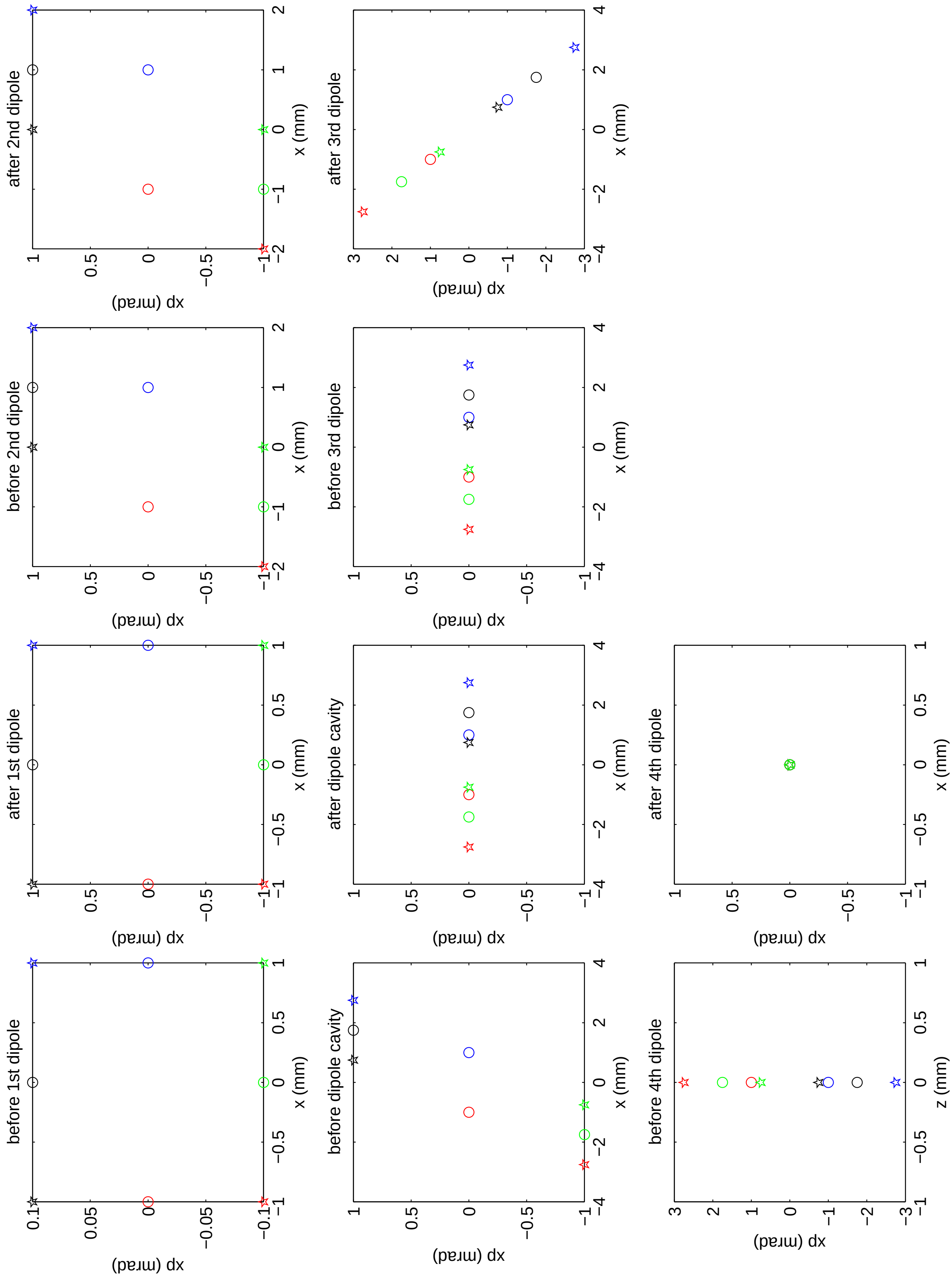

FIG. 1: group 1: x-xp.

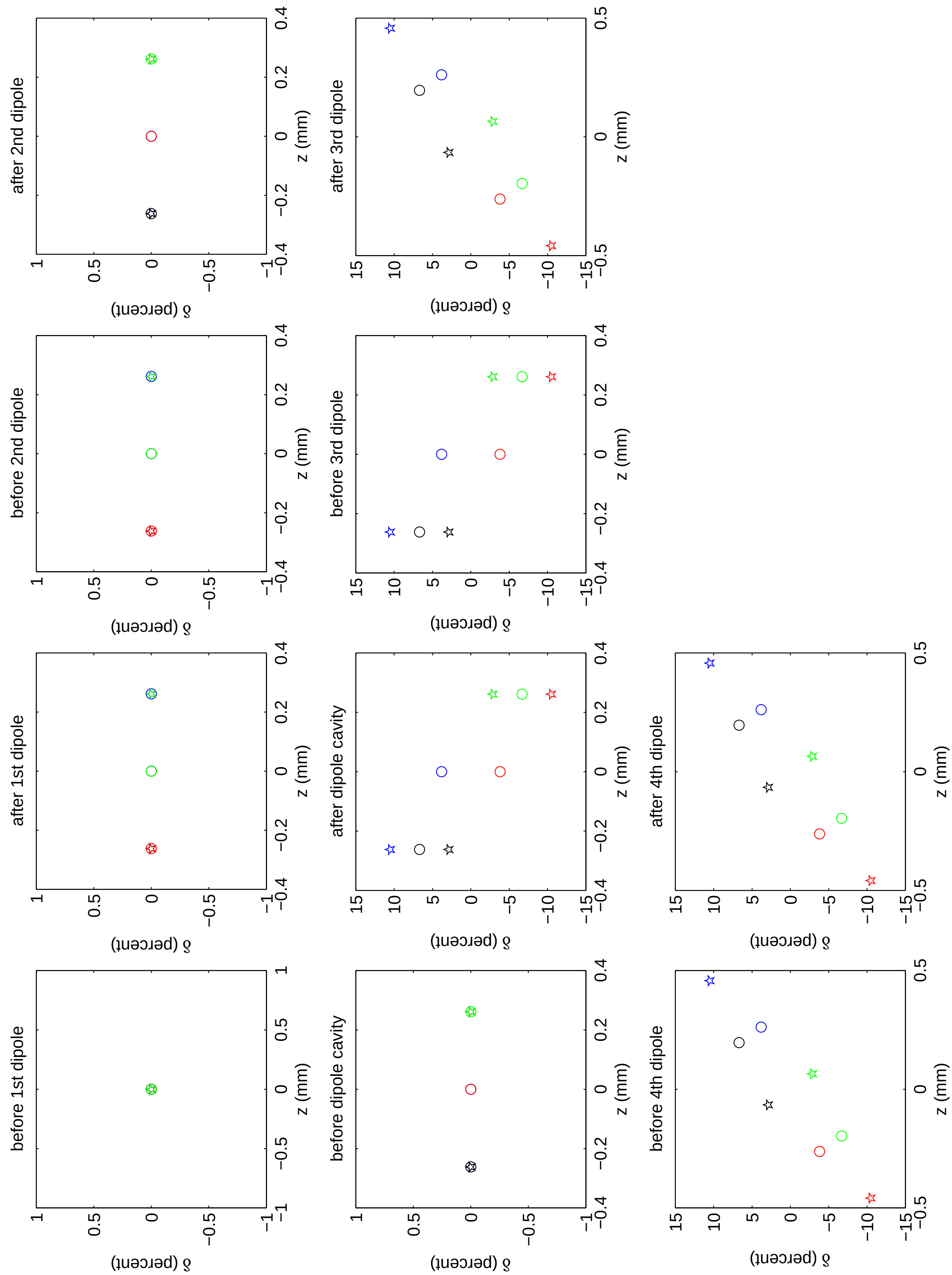

FIG. 2: group 1:z-δ.

EX--002

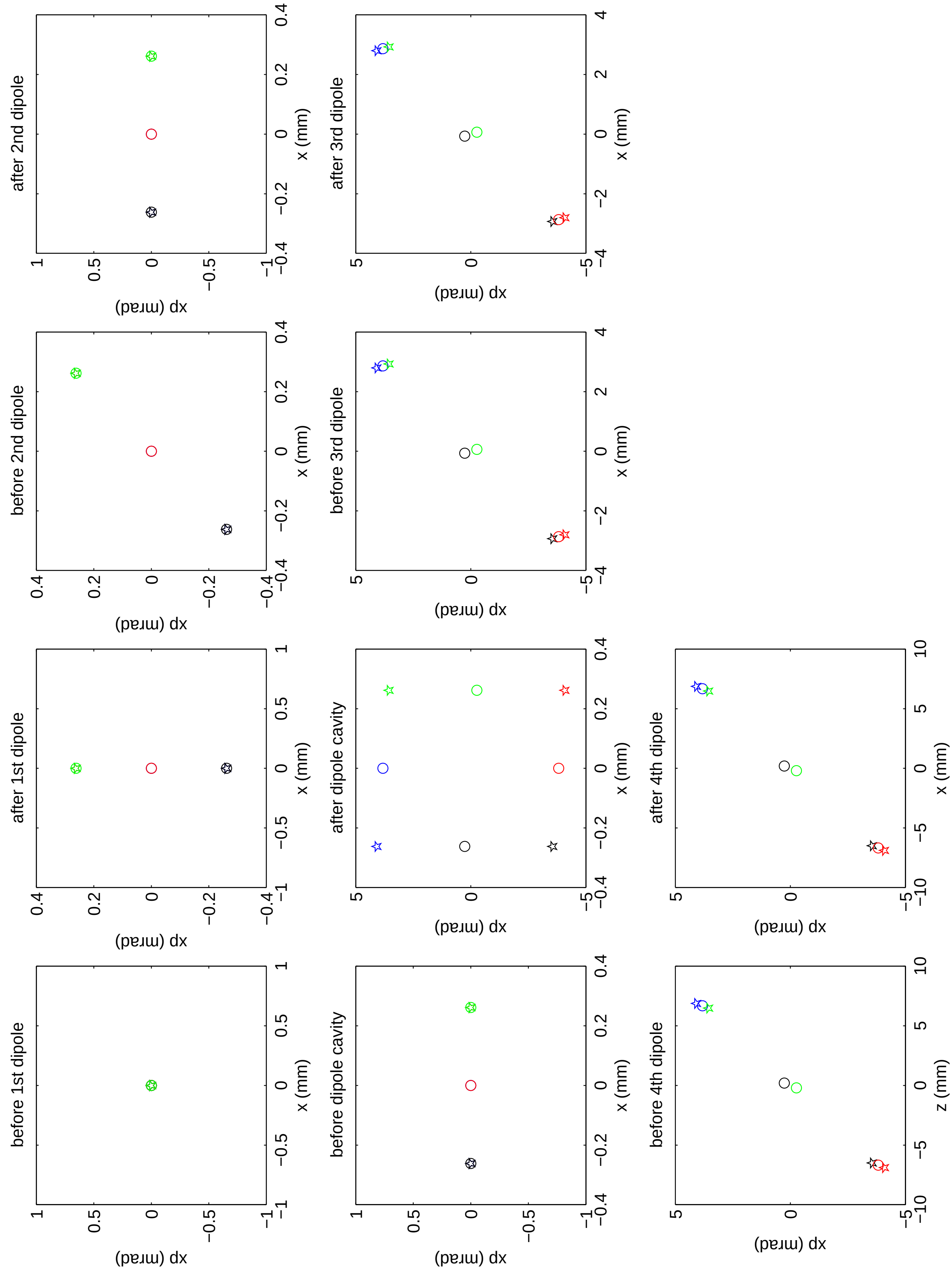

FIG. 3: group 2: x-xp.

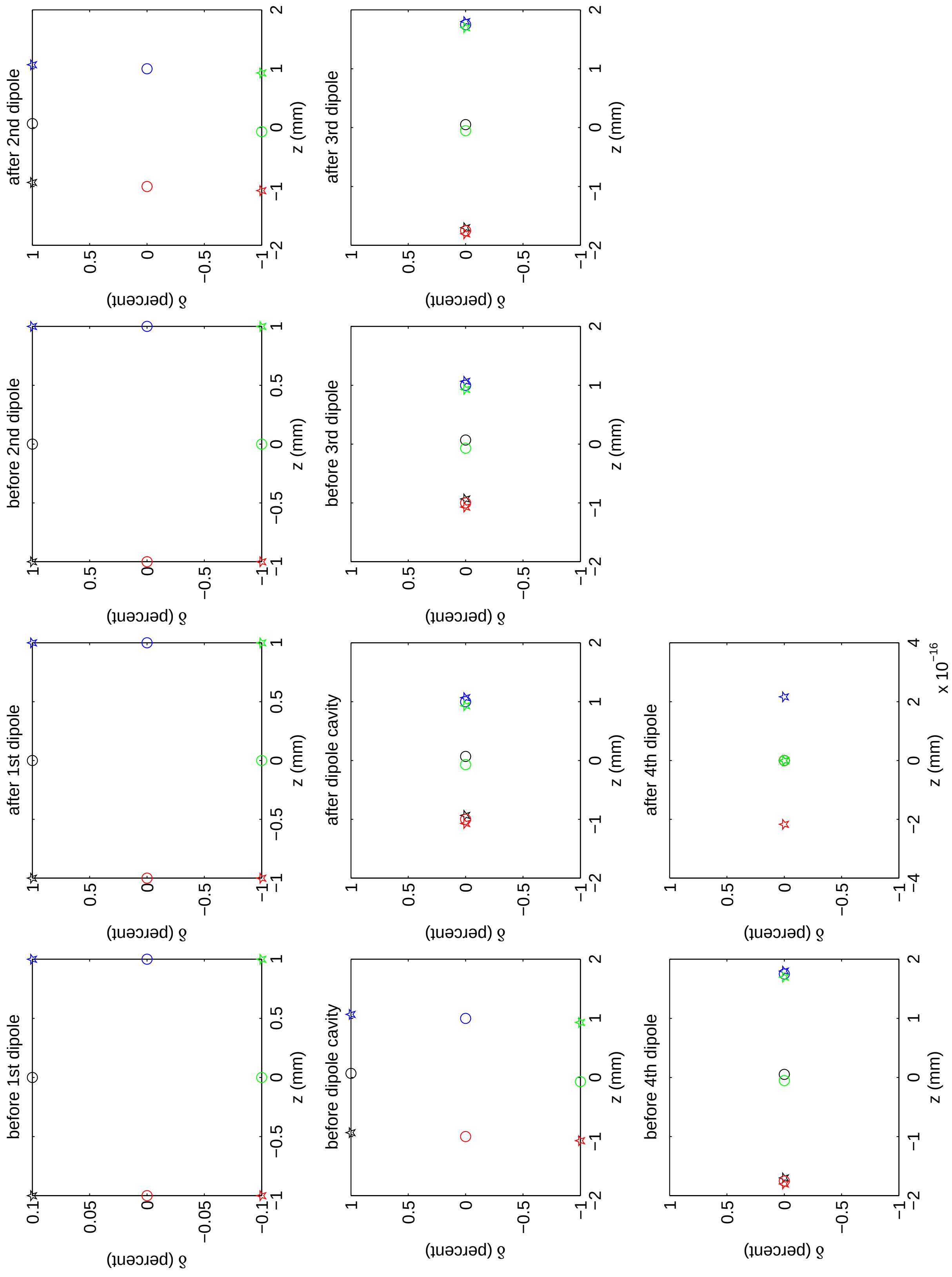

FIG. 4: group 2:z-δ.

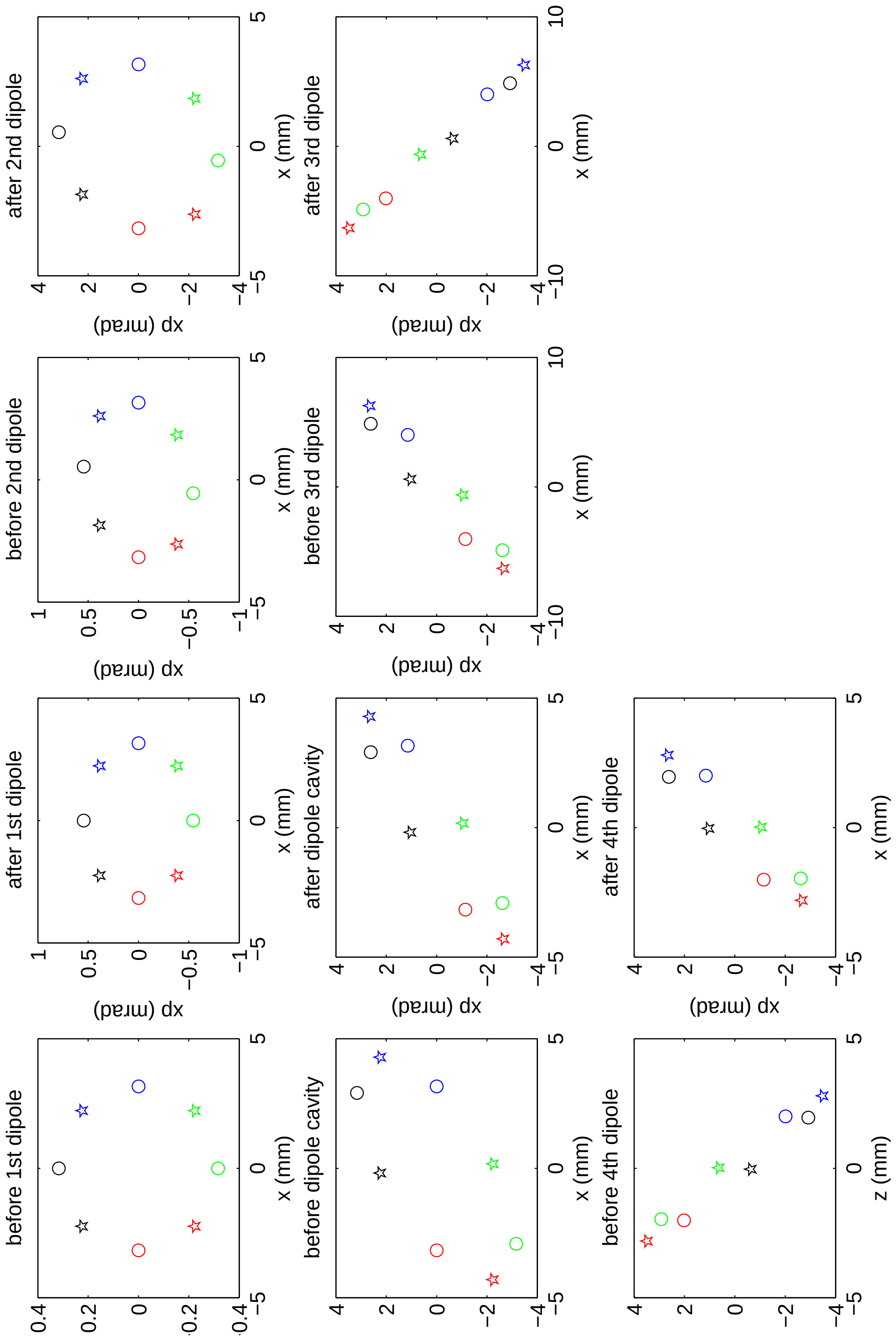

FIG. 5: group 2:x-xp.

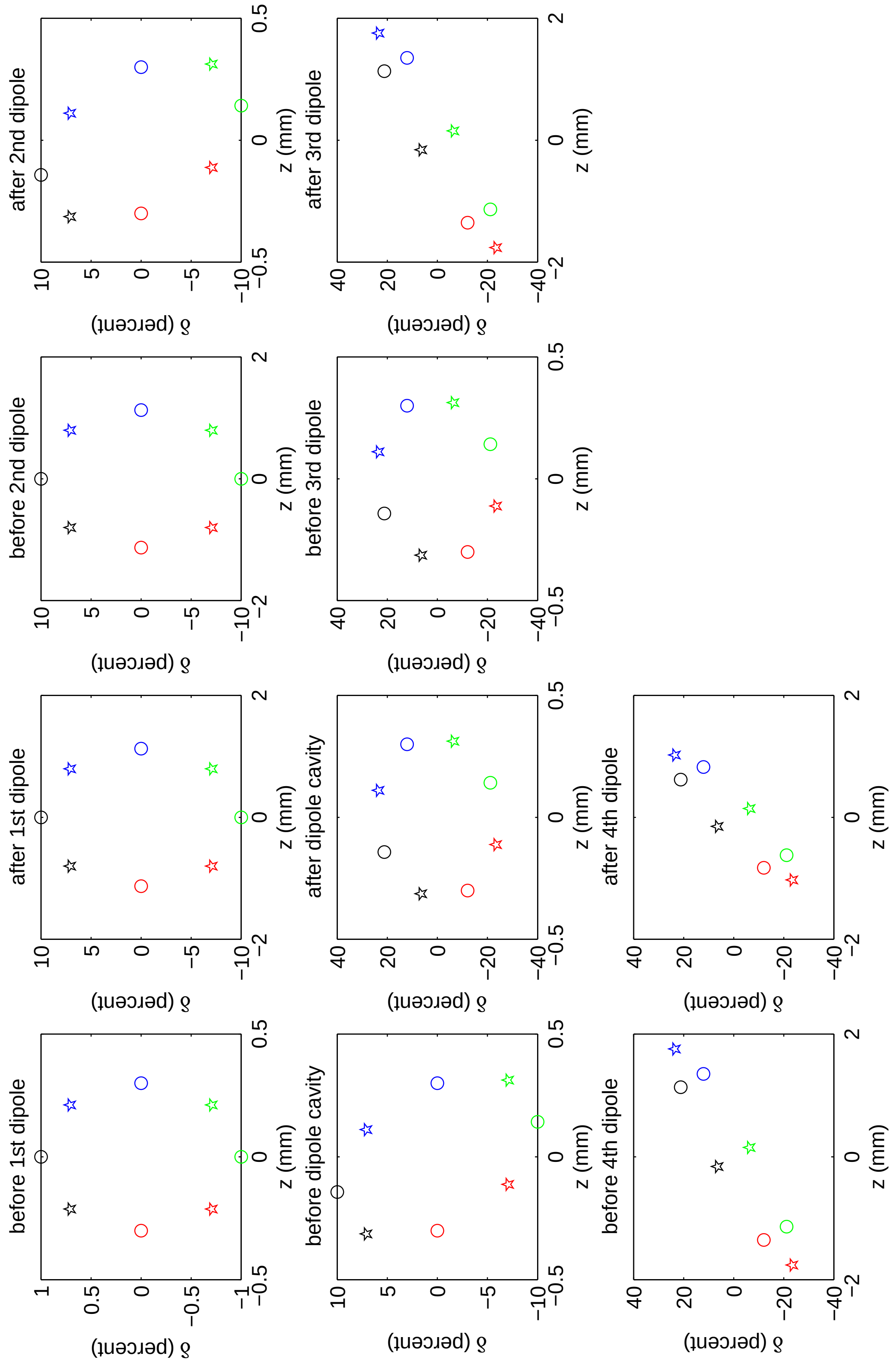

FIG. 6: group 2: z-δ.

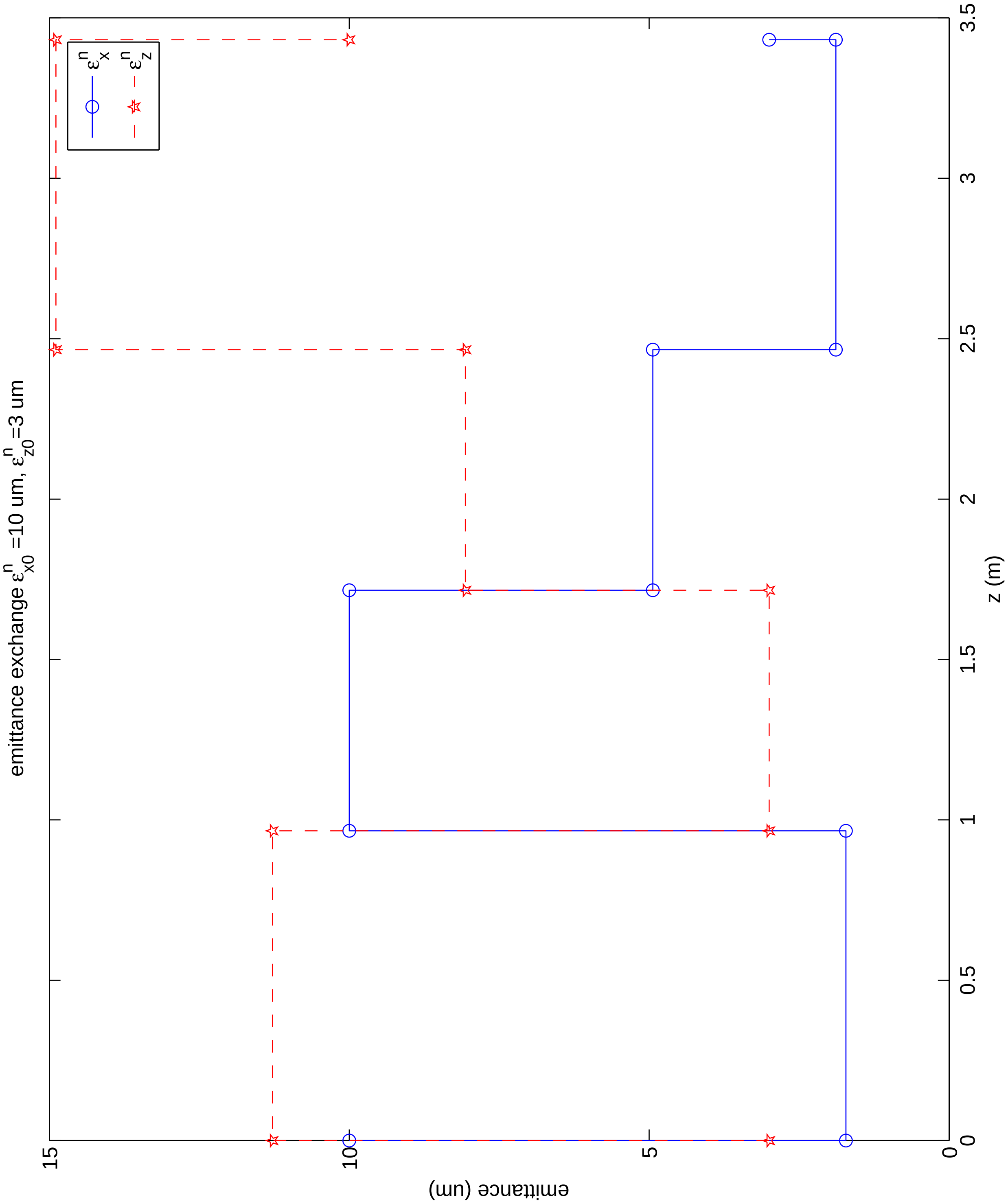

FIG. 7: group 2: ϵ_x and ϵ_z.